

On High Moments and the Spectral Norm of Large Dilute Wigner Random Matrices ^{*†‡}

O. Khorunzhiy

Université de Versailles - Saint-Quentin, Versailles, FRANCE

e-mail: oleksiy.khorunzhiy@uvsq.fr

Abstract

We consider a dilute version of the Wigner ensemble of $n \times n$ random real symmetric matrices $H^{(n,\rho)}$, where ρ denotes the average number of non-zero elements per row. We study the asymptotic properties of the spectral norm $\|H^{(n,\rho_n)}\|$ in the limit of infinite n with $s_n = O(n^{2/3})$ and $\rho_n = n^{2/3(1+\varepsilon)}$, $\varepsilon > 0$. Our main result is that the probability $\mathbf{P}\{\|H^{(n,\rho_n)}\| > 1 + xn^{-2/3}\}$, $x > 0$ is bounded for any $\varepsilon \in (\varepsilon_0, 1/2]$, $\varepsilon_0 > 0$ by an expression that does not depend on the particular values of the first several moments V_{2l} , $2 \leq l \leq 6 + \phi_0$, $\phi_0 = \phi(\varepsilon_0)$ of the matrix elements of $H^{(n,\rho)}$ provided they exist and the probability distribution of the matrix elements is symmetric. The proof is based on the detailed study of the upper bound of the moments of random matrices with truncated random variables $L_n = \mathbf{E}\{\text{Tr}(H^{(n,\rho_n)})^{2s_n}\}$.

We also consider the lower bound of L_n and show that in the complementary asymptotic regime, when $\rho_n = n^\epsilon$ with $\epsilon \in (0, 2/3]$, the fourth moment V_4 is involved into the estimates and the scale $n^{-2/3}$ at the border of the limiting spectrum should be replaced by ρ_n^{-1} .

Running title: High Moments of Large Dilute Wigner Matrices

1 Introduction

A large part of the random matrix theory is related with the studies of the spectral properties of the ensemble $\{A^{(n)}\}$ of real symmetric (or hermitian) matrices whose elements

$$(A^{(n)})_{ij} = \frac{1}{\sqrt{n}} a_{ij}, \quad 1 \leq i \leq j \leq n \quad (1.1)$$

are jointly independent random variables with zero mean value and the variance v^2 . It is common to refer to (1.1) as to the Wigner ensemble of random matrices because it was E. Wigner who studied first the eigenvalue distribution of $A^{(n)}$ in the limit of

^{*}**Acknowledgements:** The financial support of the research grant ANR-08-BLAN-0311-11 "Grandes Matrices Aléatoires" (France) is gratefully acknowledged

[†]**Key words:** random matrices, Wigner ensemble, dilute random matrices

[‡]**MSC:** 15A52

infinite n in applications to the spectral theory of heavy nuclei [27]. Random matrices of finite dimensions has been studied earlier in mathematical statistics due to their relations with the multicomponent analysis.

The semicircle law proved by E. Wigner for the limiting eigenvalue distribution of $A^{(n)}$ [27] has been then improved and generalized in many directions, in particular by relaxing the Wigner's conditions on the probability distribution of the matrix elements a_{ij} [6, 18], or by the studies of different random matrix ensembles generalizing the form of (1.1) [15], or by getting more precise information on the behavior of the extremal eigenvalues of $A^{(n)}$ and related ensembles [1, 5, 4].

At the same period, being motivated by the universality conjecture on the level repulsion in the spectra of heavy atomic nuclei [19], a strong interest to the local properties of the eigenvalue distribution of $A^{(n)}$ has lead to a number of powerful and deep results (see the monograph [16] and references therein). The breakthrough results in the studies of the local properties of the eigenvalue distribution at the border of the limiting spectrum of the random matrices of the Wigner ensemble has been obtained in paper [26], where the eigenvalue distribution of random matrices (1.1) has been studied for the first time on the local scale, i.e. on the distances of the order $n^{-2/3}$. This is done in the frameworks of the moment method, in the asymptotic regime when the order of the moments is proportional to $n^{2/3}$.

One of the possible generalizations of the Wigner ensemble (1.1) is given by the ensemble of square random matrices such that each row contains a random number of non-zero elements and the mean value of this number ρ_n is a function of the matrix dimension n . Following the statistical mechanics terminology, where such models have been first considered, it is natural to refer to this class of random matrices as to the sparse or dilute random matrices [17, 20]. The limiting eigenvalue distribution of dilute random matrices or related ensembles is studied in a number of publications, where, in particular, the semicircle law has been proved to be valid in the limit $n, \rho_n \rightarrow \infty$ [11]. The spectral properties at the edges of the limiting spectra has been studied in papers [8, 12] but the local asymptotic regime has not been reached there. In the present paper we present some results in this direction.

It should be noted that the local properties of the eigenvalue distribution of a kind of dilute random matrices is studied in [25]. However, the ensemble considered there is related with the regular trees and is different from the ensembles we study.

2 Main results

Let us consider a two-parameter family of real symmetric random matrices $\{H^{(n,\rho)}\}$ whose elements are determined by equality

$$\left(H^{(n,\rho)}\right)_{ij} = a_{ij} \eta_{ij}^{(n,\rho)}, \quad 1 \leq i \leq j \leq n, \quad (2.1)$$

where $\mathcal{A} = \{a_{ij}, 1 \leq i \leq j\}$ is an infinite family of jointly independent identically distributed random variables and $\mathcal{Y}_n = \{\eta_{ij}^{(n,\rho)}, 1 \leq i \leq j \leq n\}$ is a family of jointly independent between themselves random variables that are also independent from \mathcal{A} . We denote by $\mathbf{E} = \mathbf{E}_n$ the mathematical expectation with respect to the measure $\mathbf{P} = \mathbf{P}_n$ generated by $\mathcal{A} \cup \mathcal{Y}_n$.

We assume that the probability distribution of random variables a_{ij} is symmetric and denote their even moments by $V_{2l} = \mathbf{E}(a_{ij})^{2l}$, $l \geq 1$ with $V_2 = v^2 = 1/4$.

Random variables $\eta_{ij}^{(n,\rho)}$ are proportional to the Bernoulli ones,

$$\eta_{ij}^{(n,\rho)} = \frac{1}{\sqrt{\rho}} \begin{cases} 1 - \delta_{ij}, & \text{with probability } \rho/n, \\ 0, & \text{with probability } 1 - \rho/n, \end{cases} \quad (2.2)$$

where δ_{ij} is the Kronecker δ -symbol. Let us note that the random $n \times n$ real symmetric matrix $Y^{(n,\rho)}$ with elements $(Y^{(n,\rho)})_{ij} = \sqrt{\rho} \eta_{ij}^{(n,\rho)}$ represents the adjacency matrix of the Erdős-Rényi random graphs [2].

Our main result is related with the asymptotic behavior of the maximal in the absolute value eigenvalue of $H^{(n,\rho)}$,

$$\lambda_{\max}^{(n,\rho)} = \|H^{(n,\rho)}\| = \max_{1 \leq k \leq n} |\lambda_k(H^{(n,\rho)})|,$$

in the limit when n and ρ tend to infinity.

Theorem 2.1. *Let the probability distribution of a_{ij} be such that for some $\phi > 0$ the moment $V_{12+2\phi} = \mathbf{E}|a_{ij}|^{12+2\phi}$ exists. Then for any sequence $\rho_n = n^{2/3(1+\varepsilon)}$ with $\varepsilon > \frac{3}{6+\phi}$, the limiting probability*

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left\{ \lambda_{\max}^{(n,\rho_n)} \geq \left(1 + \frac{x}{n^{2/3}}\right) \right\} \leq \mathcal{G}(x), \quad x > 0 \quad (2.3)$$

admits the universal bound in the sense that $\mathcal{G}(x)$ does not depend on the values of V_{2l} with $2 \leq l \leq 6 + \phi$ and $V_{12+2\phi}$. One can take, in particular $\mathcal{G}(x) = \exp\{-Cx^{3/2}\}$ with a constant C .

Assuming more about the probability distributions of a_{ij} , one can relax the restriction on the range of ε of Theorem 2.1. The following statement is true.

Theorem 2.2. *Let $\tilde{a}_{ij}, 1 \leq i \leq j$ be independent identically distributed bounded random variables, $|\tilde{a}_{ij}| \leq U$ such that their probability distribution is symmetric. Then the maximal eigenvalue $\tilde{\lambda}_{\max}^{(n,\rho)} = \lambda_{\max}(\tilde{H}^{(n,\rho)})$ determined by (2.1) with the help of (2.2) admits the same asymptotic bound as (2.3)*

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left\{ \tilde{\lambda}_{\max}^{(n,\rho_n)} \geq \left(1 + \frac{x}{n^{2/3}}\right) \right\} \leq \mathcal{G}(x), \quad x > 0 \quad (2.4)$$

for $\rho_n = n^{2/3(1+\varepsilon')}$ with any given $\varepsilon' \in (0, 1/2]$.

Remark. One can prove the analogs of Theorems 2.1 and 2.2 in the case when the random matrices $H^{(n,\rho_n)}$ (2.1) are hermitian with $\eta_{ij}^{(n,\rho_n)}$ determined as before (2.2). In this case the upper bound $\mathcal{G}(x)$ can be slightly diminished. This difference disappears in the asymptotic regime of infinite n and ρ_n such that $\rho_n \ll n^{2/3}$. We discuss this issue in more details in Section 7.

Let us point out that Theorem 2.1 is in agreement with the results of paper [9], where the existence of the moment $V_{12+2\phi}$ with any positive $\phi > 0$ is proved to be sufficient for the upper bound (2.3) to hold for $\rho_n = n$, i.e. for the maximal value of

$\varepsilon = 1/2$ when the dilute random matrices coincide with those of the Wigner ensemble. In paper [10] the arguments are presented showing that the condition $\mathbf{E}|a_{ij}|^{12} < +\infty$ can be the necessary one for the upper bound (2.3) to be true for $\lambda_{\max}^{(n,n)}$.

We prove Theorems 2.1 and 2.2 basing on the detailed study of the moments

$$L_{2s}^{(n,\rho)} = \mathbf{E} \operatorname{Tr} \left(H^{(n,\rho)} \right)^{2s} = \mathbf{E} \left\{ \sum_{i_0, i_1, i_2, \dots, i_{2s-1}=1}^n H_{i_0 i_1}^{(n,\rho)} H_{i_1 i_2}^{(n,\rho)} \dots H_{i_{2s-1} i_0}^{(n,\rho)} \right\} \quad (2.6)$$

in the limit of infinite n, ρ and s . To study the case described by Theorem 2.1, we need to truncate the random variables a_{ij} . This makes the proofs of Theorems 2.1 and 2.2 very similar.

3 Even walks and classes of equivalence

Given a number $U_n > 0$, we introduce the truncated random variables

$$\hat{a}_{ij} = \hat{a}_{ij}^{(n)} = \begin{cases} a_{ij}, & \text{if } |a_{ij}| \leq U_n \\ 0, & \text{otherwise} \end{cases}$$

and consider the moments (2.6) of corresponding random matrices $\hat{H}^{(n,\rho)}$. Following E. Wigner's idea, it is natural to consider the right-hand side of (2.6) as the weighted sum

$$\mathbf{E} \operatorname{Tr} \left(\hat{H}^{(n,\rho)} \right)^{2s} = \sum_{I_{2s} \in \mathcal{I}_{2s}(n)} \hat{\Pi}_a(I_{2s}) \Pi_\eta(I_{2s}), \quad (3.1)$$

where the sequence $I_{2s} = (i_0, i_1, \dots, i_{2s-1}, i_0), i_k \in \{1, 2, \dots, n\}$ is regarded as a closed path (or trajectory) of $2s$ steps (i_{t-1}, i_t) with the discrete time $t \in [1, 2s]$. The weights $\hat{\Pi}_a(I_{2s})$ and $\Pi_\eta(I_{2s})$ are naturally determined as the average values of the products of corresponding random variables,

$$\hat{\Pi}_a(I_{2s}) = \mathbf{E} \hat{a}_{i_0 i_1} \dots \hat{a}_{i_{2s-1} i_0}, \quad \Pi_\eta(I_{2s}) = \mathbf{E} \eta_{i_0 i_1} \dots \eta_{i_{2s-1} i_0}.$$

Here and below, we omit the superscripts in $\eta_{ij}^{(n,\rho)}$ when no confusion can arise.

The proof of Theorem 2.1 is based on the following technical result.

Theorem 3.1. *Under conditions of Theorem 2.1, the following inequality holds,*

$$\limsup_{n \rightarrow \infty} \mathbf{E} \operatorname{Tr} \left(\hat{H}^{(n,\rho_n)} \right)^{2s_n} < +\infty, \quad (3.2)$$

where $s_n = \lfloor \chi n^{2/3} \rfloor$ with $\chi > 0$ and $\rho_n = n^{2/3(1+\varepsilon)}$.

We prove Theorem 3.1 with the help of the detailed study of the sum (3.1) that is based on the modified version [9] of the general approach proposed by Ya. Sinai and A. Soshnikov [22, 23, 26] for the study of high moments of large Wigner random matrices. The study of dilute random matrices is more complicated than the study of the standard Wigner ensemble. Thus, a number of essential improvements of the technique developed in [9] is introduced below. Also, some modifications of the general approach is proposed that make the computations shorter and easier.

3.1 Trajectories, walks and graphs

In this subsection we briefly repeat, with necessary modifications, the definitions of [9, 14, 26]. The notions we are going to determine are simple to illustrate but require some work to be introduced rigorously. To get an immediate image of the matter, the reader can pass to the example of the walk we give below and the descriptions of the corresponding diagrams depicted on the figure of the present section.

Given a trajectory I_{2s} , we can write that $I_{2s}(t) = i_t$ and consider a subset $\mathcal{U}(I_{2s}; t) = \{I_{2s}(t'), 0 \leq t' \leq t\} \subseteq \{1, 2, \dots, n\}$. We denote by $|\mathcal{U}(I_{2s}; t)|$ its cardinality. Each I_{2s} generates a *walk* $W_{2s} = W(I_{2s}) = \{w(t), 0 \leq t \leq 2s\}$ that we determine as a sequence of symbols from an ordered alphabet, say $\{\alpha_1, \alpha_2, \dots\}$, with the help of the following recurrence rules:

- 1) $w(0) = \alpha_1$;
- 2) if $I_{2s}(t+1) \notin \mathcal{U}(I_{2s}; t)$, then $w(t+1) = \alpha_{|\mathcal{U}(I_{2s}; t)|+1}$;
if there exists $t' \leq t$ such that $I_{2s}(t+1) = I_{2s}(t')$, then $w(t+1) = w(t')$.

In general words, the walk w_{2s} "forgets" about the particular values of $I_{2s}(t)$. For example, the trajectory $I'_{16} = (5, 2, 7, 9, 7, 1, 2, 7, 9, 7, 2, 7, 2, 1, 7, 2, 5)$ produces the walk

$$W'_{16} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_3, \alpha_5, \alpha_2, \alpha_3, \alpha_4, \alpha_3, \alpha_2, \alpha_3, \alpha_2, \alpha_5, \alpha_3, \alpha_2, \alpha_1) \quad (3.3)$$

and there is a number of trajectories that also produce W'_{16} .

Sometimes we will use greek letters $\alpha, \beta, \gamma, \dots$ instead of the symbols from the ordered alphabet $\{\alpha_1, \alpha_2, \dots\}$. We say that the pair $(w(t-1), w(t))$ represents the t -th step of the corresponding walk W_{2s} and that α_1 represents the *root* of the walk W_{2s} .

We say that the trajectories I_{2s} and \tilde{I}_{2s} are equivalent if $W(I_{2s}) = W(\tilde{I}_{2s})$. Given W_{2s} , we denote by \mathcal{C}_W the corresponding class of equivalence. It is clear that

$$|\mathcal{C}_W| = n(n-1) \cdots (n-u+1), \quad (3.4)$$

where $u = |\mathcal{U}(I_{2s}; 2s)|$ for any $I_{2s} \in \mathcal{C}_W$.

Given W_{2s} , we can determine the *multigraph* of the walk $g(W_{2s}) = (\mathcal{V}_g, \mathcal{E}_g)$ with the set \mathcal{V}_g of u vertices labelled by $\alpha_1, \dots, \alpha_u$; the oriented edge exists $(\beta, \gamma) \in \mathcal{E}_g$ if the step (β, γ) is present in W_{2s} . We attribute to each of $2s$ edges $e \in \mathcal{E}_g$ the corresponding instant of time t . We define the current multiplicity of the couple of vertices $\{\beta, \gamma\}$ up to the instant t by the following variable

$$\mathcal{M}_W^{(\{\beta, \gamma\})}(t) = \#\{t' \in [1, t] : (w(t'-1), w(t')) = (\beta, \gamma) \text{ or } (w(t'-1), w(t')) = (\gamma, \beta)\}.$$

The probability law of \hat{a}_{ij} being symmetric, the non-zero contribution to (3.2) comes from the closed trajectories I_{2s} such that in the corresponding graph of the walk of I_{2s} each couple $\{\alpha, \beta\}$ has an even multiplicity $\mathcal{M}_W^{(\{\alpha, \beta\})}(2s) = 0 \pmod{2}$. We refer to the walks of such trajectories as to the *even closed walks* and denote by \mathcal{W}_{2s} the set of all possible even closed walks of $2s$ steps. In what follows, we consider the even closed walks only and refer to them simply as to the walks.

3.2 Self-intersections, primary and imported cells

Given $W_{2s} \in \mathcal{W}_{2s}$, we say that the instant of time t is *marked* if the couple $\{\alpha, \beta\} = \{w(t-1), w(t)\}$ has an odd multiplicity at the instant t , $\mathcal{M}_W^{\{\alpha, \beta\}}(t) = 1(\text{mod } 2)$. We also say that corresponding edge $e(t) \in \mathcal{E}_g$ and step of W_{2s} are marked. All other steps and edges are called the *non-marked* ones.

It is clear that any even closed walk of $2s$ steps generates a sequence θ_{2s} of s marked and s non-marked instants that can be naturally regarded as a binary sequence of 0's and 1's. In fact, this sequence θ_{2s} is known to encode a Dyck path. So, the set of all sequences Θ_{2s} is in one-by-one correspondence with the set of all half-plane rooted trees $T_s \in \mathcal{T}_s$ constructed with the help of s edges. The cardinality of \mathcal{T}_s being given by the Catalan number $|\mathcal{T}_s| = \frac{(2s)!}{s!(s+1)!} = \tau_s$, we allow ourself to refer to the elements of \mathcal{T}_s as to the Catalan trees.

Regarding a vertex $\beta \in \mathcal{V}_g$, let us denote by $1 \leq t_1^{(\beta)} \leq t_2^{(\beta)} \leq \dots \leq t_N^{(\beta)}$ all marked instants of time such that $W_{2s}(t_j^{(\beta)}) = \beta$. We say that the N -plet $(t_1^{(\beta)}, \dots, t_N^{(\beta)})$ represents the *marked arrival instants* at the vertex β . For any non-root vertex β , we have $N = N_\beta \geq 1$. If $N_\beta = k$, we say that β is the vertex of k -fold self-intersection [22]. In general, we denote by $\kappa(\gamma)$ the *self-intersection degree* N_β of γ . If $\gamma = \alpha_1$ is the root vertex, we set $\kappa(\alpha_1) = N_{\alpha_1} + 1$.

If $\kappa(\beta) = 2$, then we say that β is the vertex of a simple self-intersection. If the walk is such that at the second marked arrival instant $t_2^{(\beta)}$ there is at least one couple $\{\beta, \gamma_j\}$ passed odd number of times, $\mathcal{M}_W^{\{\beta, \gamma_j\}}(t_2^{(\beta)}) = 1(\text{mod } 2)$, then β is referred to as to the vertex of *open self-intersection* and $t_2^{(\beta)}$ is the instant of open self-intersection [23].

Let us consider the notions of primary and imported cells introduced in [14]. Given a walk $W_{2s} = (w(0), w(1), \dots, w(2s)) \in \mathcal{W}_{2s}$, we determine the following procedure of reduction \mathcal{P} : find an instant of time t such that the step $(t-1, t)$ is marked and $w(t-1) = w(t+1)$; if such t exists, then construct a new walk $W_{2s-2}^{(1)} = \mathcal{P}(W_{2s})$ by the sequence

$$W_{2s}^{(1)} = (w(0), w(1), \dots, w(t-1), w(t+2), \dots, w(2s))$$

where two steps $(t-1, t)$ and $(t, t+1)$ are eliminated from W_{2s} .

Apply the reduction procedure \mathcal{P} to $W_{2s}^{(1)}$, get $W_{2s-4}^{(2)}$, and then repeat this action as many times as possible. Denote the resulting walk by $\bar{W}_{2s'}$, $0 \leq s' \leq s$. Regarding the multigraph $g(\bar{W}_{2s'}) = (\bar{\mathcal{V}}_g, \bar{\mathcal{E}}_g)$ as a subgraph of $g(W_{2s}) = (\mathcal{V}_g, \mathcal{E}_g)$, we keep the instants t' of edges $e(t') \in \bar{\mathcal{E}}_g(\bar{W}_{2s'})$ as they were of the edges of $e \in \mathcal{E}(W_{2s})$.

For any vertex $\beta \in \mathcal{V}_g$, we refer to the marked arrival edges at β as to the *primary cells* at β . Let us consider the reduced walk $\bar{W}_{2s'}$ and a vertex $\gamma \in \bar{\mathcal{V}}_g(\bar{W}_{2s'})$, the first arrival instant t at γ and the instant t' of the non-marked arrival at γ . Let us consider the Catalan tree $T(\bar{W}_{2s'})$ and compare the position of its vertices determined by t and t' . If these vertices do not coincide, we say that the instant t' represents an *imported cell* at $\gamma \in \mathcal{V}_g(W_{2s})$.

It should be noted here that the class of reduced walks we consider \bar{W}_{2s} coincides with the family of non-backtracking walks first used in relations with the random matrix theory in [24] and then developed in [3] and [25] and other papers.

3.3 BTS-instants

In paper [14], the notion of the instant of broken tree structure, in abbreviated form *BTS-instant* was introduced and it was shown that the number of imported cells at β is bounded by the number $2\kappa(\beta) + K$, where K is the number of *remote BTS-instants* performed by the walk. Let us recall here that the pair of steps $(t' - 1, t')$ and $(t'', t'' + 1)$ represents the BTS-instant of W_{2s} if both of $(t' - 1, t')$ and $(t'', t'' + 1)$ are present in the reduced walk \bar{W}_{2s} , are consecutive and $(t' - 1, t')$ is marked and $(t'', t'' + 1)$ is non-marked. One can say that, in certain sense, the step $(t'', t'' + 1)$ closes the marked edge that is not proper to it. Sometimes we will say simply that the couple $(t', t'' + 1)$ represent the BTS-instant, when no confusion can arise.

Finally, let us define the *exit cluster* $\mathcal{D}(\beta)$ of the vertices β as the set of vertices $\gamma_j, 0 \leq j \leq d$ such that the edges $(\beta, \gamma_i) \in \mathcal{E}_g$ are marked. We will also say that the corresponding set of edges $\Delta(\beta) = \{(\beta, \gamma_j), 0 \leq j \leq d\}$ represents the *exit edge cluster* of W_{2s} of cardinality d . Regarding the primary and imported arrival instants at β with non-empty exit edge cluster $\Delta(\beta)$, we see that the set $\Delta(\beta)$ is separated into subsets of edges Δ_j such that the corresponding edges of the Catalan tree $T(W_{2s})$ have the same parent for any two edges from the same subset Δ_j and the edges from different subsets Δ_j and $\Delta_{j'}$ have different parents. These subsets are attributed in natural and unique way to the primary and imported arrival instants at β that explains their role in the studies of the maximal exit degree of the vertices of the walks.

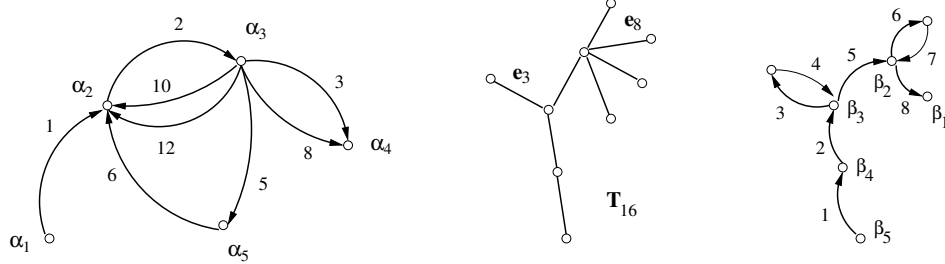


Figure 1: Graph of the walk W'_{16} , its tree $T(W')$ and a part of its Catalan structure $\theta^{(8)}$

We complete this subsection with the definition of the *short BTS-instant*. We say that the couple $(t' - 1, t')$, $(t'', t'' + 1)$ represents the short BTS-instant of the walk W_{2s} if this couple is the BTS-instant and the marked edge $e(t')$ joins the vertices α, β such that there exists at least one marked edge $e(t) = \{\alpha, \beta\}$ with $t < t'$.

Let us consider the example walk W'_{16} given by the sequence (3.3). The graph $g(W'_{16})$ is depicted on Figure 1. We see that $\kappa(\alpha_1) = \kappa(\alpha_3) = \kappa(\alpha_5) = 1$, $\kappa(\alpha_2) = 4$ and $\kappa(\alpha_4) = 2$. There is one open simple self-intersection (1,6) in the walk. The vertex α_3 has one primary cell at $t = 2$ and one imported cell at $t = 7$. Regarding the Catalan tree $T(W'_{16})$, we see that the edges e_3 and e_8 have different parents in T . This means that the edge e_8 is imported to the vertex α_3 and explains the term of imported cell. According to the definitions, all other vertices of $g(W'_{16})$ have no imported cells. The couples (6,7) and (12,13) represent the BTS-instants of W'_{16} with (12,13) being the short one.

3.4 Classification of vertices and edges

Given a walk W_{2s} , let us introduce one more multigraph $\tilde{g}(W_{2s}) = (\mathcal{V}_g, \tilde{\mathcal{E}}_g)$ such that $\tilde{e} \in \tilde{\mathcal{E}}_g$ contains the marked edges of \mathcal{E}_g only. Certainly, we keep the time labels of \tilde{e} as they were in \mathcal{E}_g . Having an integer k_0 , we consider all vertices β such that their self-intersection degree $\kappa(\beta) \leq k_0$ and say that they are the μ -vertices. We denote the set of μ -vertices by $\mathcal{V}_g^{(\mu)}$.

The vertices γ with $\kappa(\gamma) \geq k_0 + 1$ are referred to as the ν -vertices. We denote the set of ν -vertices by $\mathcal{V}_g^{(\nu)}$. Regarding Figure 1 and taking for instance $k_0 = 3$, we see that there are four μ -vertices (including the root vertex α_1) and one ν -vertex α_2 .

Let us classify the edges of $\tilde{g}(W_{2s})$. Regarding $\{\beta, \gamma\} \in \tilde{\mathcal{E}}_g$, we see that one of the following three cases can occur.

1. The vertices β and γ are both μ -vertices. In this case we consider the last passage of $\{\beta, \gamma\}$ in the chronological order, assume (γ, β) , and say that the corresponding oriented edge (γ, β) is the μ -edge that joins β and γ .

2. Let the edge $\{\beta, \gamma\} \in \tilde{\mathcal{E}}_g$ be such that β is the μ -vertex and γ is the ν -vertex. We say that all edges of the form (β, γ) are ν -edges. Regarding the last passage among $p + 1$ edges of the form (γ, β) , we determine the corresponding oriented edge (if it exists) as the μ -edge. In this case we say that β and γ are joined by a μ -edge (γ, β) .

3. If β is the ν -vertex, we say that all edges of the form (γ_j, β) of $\tilde{\mathcal{E}}_g$ are the ν -edges. If β and γ are both ν -vertices, then all edges of the form $\{\beta, \gamma\}$ are the ν -edges. We consider the last passage of $\{\beta, \gamma\}$ and say that the corresponding oriented ν -edge is the *base ν -edge*.

Regarding all other edges of $\tilde{\mathcal{E}}_g$ that are nor μ -edges neither ν -edges, we say that they represent the p -edges of the graph $(\mathcal{V}_g, \tilde{\mathcal{E}}_g)$. It is helpful to think that the μ -, p -, and ν -edges are colored in, say, blue, green and black colors, respectively. Clearly, the green edges $\{\alpha, \beta\}$ exist if and only if there exists one and unique blue edge of the form (α, β) or (β, α) . In this case we say that these green edges are attributed to the blue one.

To complete the classification, we perform the following action that we name the *re-coloring procedure*. When all edges of g are classified into the blue, black and green ones, we look for the last in the chronological order μ -vertex β' such that there is no blue μ -edge of the form (α, β') . This can happen when all oriented edges (α_j, β') are the green ones placed over the blue edges (β', α_j) . Then we choose the last ever green arrival at β' and change the color of the corresponding green edge (α', β') to the blue one. The blue edge (β', α') is then re-colored from the blue to the green color.

Let us stress that the re-coloring procedure concerns only the edges that join the pairs of μ -vertices. Another important thing is that the vertex α' loses one blue arrival edge. It is not hard to see that in such a situation there are some more μ -edges of the form (γ_i, α') . It requires some work to prove that the sequence of the re-coloring procedures terminates after a finite number of steps. This can be done in the spirit of the arguments used in [14] and we do not present the complete proof here.

After all of the re-colorings of green edges is done, any μ -vertex γ has at least one blue arrival edge of the form (β, γ) . If the μ -vertex γ is such that there are m blue edges of the form (γ_j, β) then we say the β is of the μ -self-intersection degree m and write $\kappa_\mu(\beta) = m$. Certainly, if γ is the ν -vertex, then $\kappa_\nu(\gamma) = \kappa(\gamma)$. Looking at the Figure 1, we see that the vertex α_3 gets the arrival blue edge η_3 after the re-coloring procedure.

The blue edges are determined as the last arrival instants in order to get the correct description of the BTS-instants. The re-coloring procedure is related with the short BTS-instants only. Their number is estimated by the number of all other BTS-instants (see Subsection 6.4). Therefore this procedure changes nothing with respect to the estimates we use below.

3.5 Classes of walks, weights and cardinalities

Given W_{2s} , we say that it belongs to the class of walks $\mathcal{C}_{k_0}(\bar{\mu}, \bar{P}, \bar{\nu})$ with values $\bar{\mu} = (\mu_2, \mu_3, \dots, \mu_{k_0})$, $\bar{P} = (p_1, p_2, \dots, p_{2k_0})$, $\bar{\nu} = (\nu_{k_0+1}, \dots, \nu_s)$ and integers μ_m , p_l and ν_k , in the cases when the graph $g(W_{2s})$ has μ_m vertices of μ -self-intersection degree m , p_l μ -edges e with multiplicity $\pi(e) = p_l$ and ν_k vertices with ν -self-intersection degree k . It is clear that the number of vertices in the graph $g(W_{2s})$ is given in this case by the following expression

$$|\mathcal{V}_g| = \sum_{m=1}^{k_0} \mu_m + \sum_{k=k_0+1}^s \nu_k. \quad (3.5)$$

The weight of the trajectory $\Pi(I_{2s})$ is naturally determined as a product of the weights $\Pi_e(\{\alpha, \beta\})$ over the edges $\{\alpha, \beta\}$ of the graph $g(W_{2s})$ with $W(I_{2s}) = W_{2s}$. However, we would to use the estimate of $\Pi(I_{2s})$ expressed in terms of the product over the vertices of $g(W_{2s})$. One of such possible expressions is given by the following statement.

Lemma 3.1. *Any trajectory I_{2s} such that $W(I_{2s}) \in \mathcal{C}_{k_0}(\bar{\mu}, \bar{p}, \bar{\nu})$ has the weight $\Pi(I_{2s})$ that can be estimated as follows,*

$$\Pi(I_{2s}) \leq \frac{1}{n^{|\mathcal{V}_g|+\sigma}} \hat{V}_2^{|\bar{\mu}|} \cdot \prod_{l=1}^{2k_0-1} \left(\frac{\hat{V}_{2+2l}}{\rho^l} \right)^{p_l} \cdot \prod_{k=k_0+1}^s \left(\frac{nU_n^{2k}}{\rho^k} \right)^{\nu_k}. \quad (3.6)$$

where we denoted $\sigma = \sum_{m=1}^{k_0} (m-1)\mu_m$.

Proof. Given a vertex α , we determine the corresponding vertex weight $\Pi_v(\alpha)$ with the help of the arriving blue and black edges. More precisely, we act as follows.

Let α be a μ -vertex with m' different arrival μ -edges that have no p -edges attributed to them and m'' other arrival μ -edges. Then the contribution of this vertex α to the estimate of Π is as follows,

$$\left(\frac{\hat{V}_2}{n} \right)^{m'} \cdot \prod_{j=1}^{m''} \left(\frac{\hat{V}_{2+2l_j}}{n\rho^{l_j}} \right)^{p_{l_j}}, \quad l_j \geq 1. \quad (3.7)$$

Then it is clear that the product over all μ -vertices gives the following expression

$$\prod_{\alpha \in \mathcal{V}_g^{(\mu)}} \Pi_v(\alpha) = \frac{\hat{V}^{|\bar{\mu}|}}{n^{|\bar{\mu}|}} \cdot \prod_{l=1}^{2k_0-1} \left(\frac{\hat{V}_{2+2l}}{\rho^l} \right)^{p_l}, \quad (3.8)$$

where we denoted $|\bar{\mu}| = \sum_{m=1}^{k_0} m\mu_m$.

If β is the ν -vertex, then all k arrival edges of the form (γ_j, β) , $j \geq 1$ are the black ones. If the vertex γ_i is the μ -vertex with l_i attributed green edges and q_i black edges (γ_i, β) , then we write the following estimate for the edge weight

$$\Pi_e(\{\beta, \gamma_i\}) \leq \frac{\hat{V}_{2+2l_1+2q_1}}{n\rho^{l_i+q_i}} \leq \frac{\hat{V}_{2+2l_i}}{n\rho^{l_i}} \cdot \frac{U_n^{2q_i}}{\rho^{q_i}} \quad (3.9)$$

that is in agreement with (3.6).

If there exists a ν -vertex γ_j such that there are q' edges of the form (γ_j, β) and q'' edges of the form (β, γ_j) , then we write the following estimate for the edge weight

$$\Pi_e(\{\beta, \gamma_j\}) \leq \frac{U_n^{2q'}}{\rho^{q'}} \cdot \frac{U_n^{2q''}}{\rho^{q''}}. \quad (3.10)$$

It is clear that (3.9) and (3.10) imply the following estimate of the product over all ν -vertices,

$$\prod_{\beta \in \mathcal{V}_g^{(\nu)}} \Pi_v(\beta) \leq \prod_{k=k_0+1}^{k_0} \left(\frac{U_n^{2k}}{\rho^k} \right)^{\nu_k}. \quad (3.11)$$

Combining (3.8) and (3.11) and taking into account relation (3.5), we get the estimate (3.6). Lemma 3.1 is proved.

Let us make the following remark with respect to the weight of ν -vertices. If $\beta \in \mathcal{V}_g^{(\nu)}$ is such that there exist the base ν -edge (γ_j, β) , then the right-hand side of (3.10) can be replaced by more strong estimate $U_n^{2q'+2q''}/(n\rho^{q'-1+q''})$. However, there can exist vertices β' such that there is no base ν -edges arriving at them. Thus, in the general case we are forced to use (3.10).

The notion of the self-intersection of even walk introduced in [22] helps to estimate the number of walks of the same class. This upper bound given below is slightly different with respect to the modification of the basic technique of [22, 23], proposed in [9].

Lemma 3.2. *Let us consider a subset $\mathcal{C}_{\theta,d,k_0}(\bar{\mu}, \bar{p}, \bar{\nu}) \in \mathcal{C}_{k_0}(\bar{\mu}, \bar{p}, \bar{\nu})$ of walks W_{2s} such that the Catalan structure of W_{2s} is given by θ and the maximal exit degree of W_{2s} is d . Then*

$$\begin{aligned} |\mathcal{C}_{\theta,d,k_0}(\bar{\mu}, \bar{p}, \bar{\nu})| &\leq \frac{1}{r!(\mu_2 - r)!} (6s\theta^*)^r \left(\frac{\tilde{s}^2}{2} \right)^{\mu_2 - r} \prod_{m=3}^{k_0} \frac{1}{\mu_m!} \left(\frac{(2k_0 s)^m}{m!} \right)^{\mu_m} \\ &\quad \times \prod_{l=1}^{2k_0-1} \frac{(sd^l)^{p_l}}{p_l!} \cdot \prod_{k=k_0+1}^s \frac{(C_1 s)^{k\nu_k}}{\nu_k!}, \end{aligned} \quad (3.12)$$

where $\theta^* = \max_t \theta(t)$, $\tilde{s} = s - \zeta_\mu^{(3)} - \zeta_p - \zeta_\nu$ with

$$\zeta_p = |\bar{p}| = \sum_{l=1}^{2k_0-1} l p_l, \quad \zeta_\nu = |\bar{\nu}| = \sum_{k=k_0+1}^s k \nu_k, \quad \zeta_\mu^{(3)} = \sum_{m=3}^{k_0} m \mu_m,$$

and $C_1 = \sup_{k \geq 1} 2k(k!)^{-1/k}$.

We prove this Lemma in Section 6.

4 Estimates from above

Slightly modifying the general scheme proposed in [26] and further improved in [21] and [9], we split the sum over trajectories (3.2) into three parts,

$$\mathbf{E} \operatorname{Tr} \left(\hat{H}^{(n,\rho)} \right)^{2s} = Z_{2s}^{(1)} + Z_{2s}^{(2)} + Z_{2s}^{(3)},$$

where $Z_{2s}^{(1)}$ is the sum over the trajectories $I_{2s} \in \mathcal{C}(W_{2s})$ such that the graphs $g(W_{2s})$ have no multiple edges, $Z_{2s}^{(2)}$ is the sum over the trajectories $I_{2s} \in \mathcal{C}(W_{2s})$ such that the graph $g(W_{2s})$ have at least one multiple edge and the maximal exit degree $g(W_{2s})$ is bounded $D \leq n^\epsilon$, and $Z_{2s}^{(3)}$ is the sum over the trajectories $I_{2s} \in \mathcal{C}(W_{2s})$ such that the graph $g(W_{2s})$ have at least one multiple edge and the maximal exit degree $D > n^\epsilon$. The value of the technical parameter $\epsilon > 0$ will be presented below.

4.1 Estimate of $Z_{2s}^{(1)}$

It is easy to see that to estimate $Z_{2s}^{(1)}$, we do not need to distinguish the μ and ν -vertices and we write down the estimate by using the μ -vertices only.

We split the sum $Z_{2s}^{(1)}$ into two parts $Z_{2s}^{(1,1)}$ and $Z_{2s}^{(1,2)}$ in dependence of whether μ_2 is less than $C_0 s^2/n$ or not, where C_0 is a constant. This partition is similar to that used in [21], but here we use it in much simpler form.

It follows from the definitions that

$$Z_{2s}^{(1,1)} = \frac{\hat{V}_2^s}{n^s} \sum_{\theta \in \Theta_{2s}} \sum_{\mu_2=0}^{C_0 s^2/n} \sum_{\bar{\mu}} \sum_{W \in \mathcal{C}_\theta(\bar{\mu})} |\mathcal{C}_W|, \quad (4.1)$$

where according to (3.4), $|\mathcal{C}_W| = n(n-1) \cdots (n - |\mathcal{V}_g| + 1)$. Taking into account equality $|\mathcal{V}_g| = s - \sum_{m=3}^s (m-1)\mu_m - \mu_2 = \hat{s} - \mu_2$, we can write that

$$\frac{(n-1) \cdots (n - |\mathcal{V}_g| + 1)}{n^s} = \frac{(n-1) \cdots (n - (\hat{s} - \mu_2) + 1)}{n^{\hat{s}-\mu_2} n^{\mu_2} n^\sigma},$$

where we denoted $\hat{s} = s - \sigma$ with $\sigma = \sum_{m=3}^s (m-1)\mu_m$.

Simple computation leads to the following important inequality [23]

$$\frac{(n-1) \cdots (n - \hat{s} + \mu_2)}{n^{\hat{s}-\mu_2}} \leq \exp \left\{ -\frac{\hat{s}^2}{2n} + \frac{\hat{s}\mu_2}{n} \right\}. \quad (4.2)$$

Taking into account that $\hat{s} \leq s \leq \chi n^{2/3}$ and using (3.12) with obvious changes, we deduce from (4.1) the upper bound

$$\begin{aligned} Z_{2s}^{(1,1)} &\leq n V_2^s e^{C_0 \chi^2} \sum_{\theta \in \Theta_{2s}} e^{-\hat{s}^2/(2n)} \sum_{\bar{\mu}} \frac{1}{\mu_2!} \left(\frac{\hat{s}^2}{2n} + \frac{6s\theta^*}{n} \right)^{\mu_2} \prod_{m=3}^s \frac{1}{\mu_m!} \left(\frac{(C_1 s)^m}{n^{m-1}} \right)^{\mu_m} \\ &\leq \frac{n}{4^s} \frac{(2s)!}{s!(s+1)!} B_s(6\chi^{3/2}) \exp\{(C_0 + 36)\chi^3 + \delta_n\}, \end{aligned} \quad (4.3)$$

where $\tilde{s} = s - \zeta_\mu^{(3)} \leq \hat{s}$,

$$B_s(z) = \frac{1}{|\Theta_{2s}|} \sum_{\theta \in \Theta_{2s}} \exp\left\{ \frac{z\theta^*}{\sqrt{s}} \right\},$$

and $\delta_n = o(1)$ in the limit $n \rightarrow \infty$.

Using the asymptotic expression for the factorials and taking into account the existence of the limit $B(z) = \lim_{s \rightarrow \infty} B_s(z)$ [13], we see that (4.3) implies the following estimate

$$\limsup_{n \rightarrow \infty} Z_{2s_n}^{(1,1)} \leq \frac{1}{\sqrt{\pi\chi^3}} B(6\chi^{3/2}) e^{(C_0+36)\chi^3}. \quad (4.4)$$

Let us consider $Z_{2s}^{(1;2)}$. In this case inequality (4.2) is of no use and there is no compensation factor $\exp\{-s^2/(2n)\}$ to neutralize the sum over μ_2 . However, large factor $\mu_2!$ serves by itself as the compensation for the sum over μ_2 .

Indeed, we can write that

$$\begin{aligned} Z_{2s}^{(1,2)} &\leq \frac{n}{4^s} \frac{(2s)!}{s!(s+1)!} \sum_{\mu_2 > C_0 s^2/n} \sum_{\bar{\mu}} \frac{1}{\mu_2!} \left(\frac{s^2}{2n} \right)^{\mu_2} \prod_{m=3}^s \frac{1}{\mu_m!} \left(\frac{(C_1 s)^m}{n^{m-1}} \right)^{\mu_m} \\ &\leq \frac{1}{\sqrt{\pi\chi^3}} \exp\{C_1 \chi^3 (1 + o(1))\} \sum_{\mu_2 > C_0 s^2/n} \frac{1}{\sqrt{2\pi\mu_2}} \left(\frac{es^2}{2\mu_2 n} \right)^{\mu_2}. \end{aligned}$$

The last series is obviously $o(1)$ as $n \rightarrow \infty$ provided $C_0 \geq e$. This conclusion together with (4.4) implies the finiteness of $Z_{2s_n}^{(1)}$ in the limit $n \rightarrow \infty$.

4.2 Estimate of $Z_{2s}^{(2)}$

Let us choose

$$k_0 = \lfloor \frac{3}{\varepsilon - \varepsilon_0} \rfloor \quad \text{and} \quad \delta = \frac{\varepsilon_0 + \varepsilon}{6}. \quad (4.6)$$

As in the previous subsection, we split $Z_{2s}^{(2)}$ into two sub-sums according to the values of the sum μ_2 .

Using (3.6) and (3.12), we can write the following estimate

$$\begin{aligned}
Z_{2s}^{(2,1)} &\leq n \hat{V}_2^s \sum_{d=1}^{n^\epsilon} \sum_{\theta \in \Theta_{2s}} \sum_{\mu_2=0}^{C_0 s^2/n} \sum_{\bar{\mu}} \exp\left(-\frac{\hat{s}^2}{2n} + s\mu_2\right) \\
&\times \frac{1}{\mu_2!} \left(\frac{\hat{s}^2}{2n} + \frac{6s\theta^*}{n}\right)^{\mu_2} \cdot \frac{1}{\mu_3!} \left(\frac{32s^3}{3n^2}\right)^{\mu_3} \prod_{m=4}^s \frac{1}{\mu_m!} \left(\frac{(2k_0 s)^m}{n^{m-1}}\right)^{\mu_m} \\
&\times \sum_{\bar{p}, \bar{\nu}}^* \prod_{l=1}^{2k_0-1} \frac{1}{p_l!} \left(\frac{s d^l}{\rho^l} \tilde{V}_{2l+2}\right)^{p_l} \cdot \prod_{k=k_0+1}^s \frac{1}{\nu_k!} \left(n \frac{(2C_1 s)^k U_n^{2k}}{v^{2k} \rho^k}\right)^{\nu_k},
\end{aligned}$$

where we denoted $\tilde{V}_{2l+2} = \hat{V}_{2l+2}/\hat{V}_2^l$ and used inequality $\hat{V}_2 \geq v^2/2$. In the last sum, the star means that $|\bar{p}| + |\bar{\nu}| \geq 1$.

Let us use the following representation,

$$n \frac{(2C_1 s)^k U_n^{2k}}{v^{2k} \rho^k} = A_n^{(k_0)} \left(\frac{C_1 n^{2\delta}}{v^2 n^{2\varepsilon/3}}\right)^{k-k_0}, \quad (4.7)$$

where

$$A_n^{(k_0)} = \left(\frac{2C_1}{v^2}\right)^{k_0} \cdot \frac{n U_n^{2k_0}}{\rho^{k_0}} = \left(\frac{2C_1}{v^2}\right)^{k_0} \cdot \frac{n^{(2/3+2\delta)k_0+1}}{n^{2/3(1+\varepsilon)k_0}}.$$

It is clear that the choice of (4.6) implies that $A_n^{(k_0)} = o(1)$ as $n \rightarrow \infty$.

Repeating computations of the previous subsection, we get the estimate

$$\begin{aligned}
Z_{2s}^{(2,1)} &\leq \frac{n^\epsilon}{\sqrt{\pi\chi^3}} e^{(C_0+36)\chi^3} B_s(6\chi^{3/2}) \left(\exp \left\{ \sum_{l=1}^{2k_0-1} \tilde{V}_{2l+2} \frac{n^\epsilon}{n^{2\varepsilon/3}} \right\} - 1^* \right) \\
&\times \left(\exp \left\{ A_n^{(k_0)} \right\} - 1^* \right),
\end{aligned}$$

where the stars at the unities mean that at least one of them is present in this expression. Then certainly, under conditions of Theorem 2.1, $Z_{2s_n}^{(2,1)} = o(1)$ as $n \rightarrow \infty$.

Let us consider $Z_{2s}^{(2,2)}$. We can write that

$$\begin{aligned}
Z_{2s}^{(2,2)} &\leq \frac{n}{4^s} \frac{(2s)!}{s!(s+1)!} \sum_{d=1}^{n^\epsilon} \sum_{\mu_2 > C_0 s^2/n} \sum_{\bar{\mu}} \frac{1}{\mu_2!} \left(\frac{s^2}{2n}\right)^{\mu_2} \prod_{m=3}^s \frac{1}{\mu_m!} \left(\frac{(2k_0 s)^m}{n^{m-1}}\right)^{\mu_m} \\
&\times \sum_{\bar{p}, \bar{\nu}}^* \prod_{l=1}^{2k_0-1} \frac{1}{p_l!} \left(\frac{s d^l}{\rho^l} \tilde{V}_{2l+2}\right)^{p_l} \cdot \prod_{k=k_0+1}^s \frac{1}{\nu_k!} \left(n \frac{(2C_1 s)^k U_n^{2k}}{v^{2k} \rho^k}\right)^{\nu_k}. \quad (4.8)
\end{aligned}$$

Repeating computations of the previous subsection, we get from (4.8) inequality

$$\begin{aligned}
Z_{2s}^{(2,2)} &\leq n^\epsilon \frac{2}{\sqrt{2\pi\chi^2 n^{1/3}}} \left(\exp \left\{ \sum_{l=1}^{2k_0-1} \tilde{V}_{2l+2} \frac{n^\epsilon}{n^{2\varepsilon/3}} \right\} - 1^* \right) \\
&\times \left(\exp \left\{ A_n^{(k_0)} \right\} - 1^* \right), \quad (4.9)
\end{aligned}$$

that shows that $Z_{2s_n}^{(2,2)} = o(1)$ as $n \rightarrow \infty$ for $\epsilon = \varepsilon/3$. Thus, $Z_{2s_n}^{(2)}$ does not contribute to the estimate of the moments (3.2) in the limit of infinite n .

4.3 Estimate of $Z_{2s}^{(3)}$

This is the most complicated part of the estimates that requires a large amount of technical work and auxiliary statements. We try to optimize the details of the proof and the number of pages needed for it and keep the maximal clarity of the argument.

Let us consider the sub-sum of $Z_{2s}^{(3)}$ that corresponds to the sum over $\bar{\mu}$ such that $\mu_2 \leq C_0 s^2/n$. The walks we consider have the maximal exit degree $D(W_{2s}) > n^{\varepsilon/3}$ and therefore there exists a vertex $\beta_0 = W_{2s}(t_0)$ such that $|\Delta(\beta_0)| = D(W_{2s})$. Let us consider the subset of walks $\mathcal{W}_{2s}^{(t_0; d)}(\mathcal{C})$ with $\mathcal{C} = \mathcal{C}(\bar{\mu}, \bar{p}, \bar{\nu}; \bar{r}; v)$, where \bar{r} stands to point out the arrival BTS-instants and v determines the rule where to go by the non-marked steps at the BTS-instants.

Denoting by L the number of the arrival cells at β_0 and by $\bar{d}_L = (d_1, \dots, d_L)$ the repartition of the d edges among these cells, we can write that

$$\sum_{\bar{d}_L} |\mathcal{W}_{2s}^{(t_0; L, \bar{d}_L)}(\mathcal{C})| \leq \frac{(d+L-1)!}{d! (L-1)!} e^{-\eta d} |\mathcal{W}_{2s}^{(t_0, L)}(\mathcal{C})|. \quad (4.10)$$

This inequality is proved in Section 6 (see corollary of Lemma 6.5). As we will see below, the presence of L arrival cells at the vertex β_0 impose certain conditions on the classes \mathcal{C} , that helps to compensate the factorial expressions of the right-hand side of (4.10).

The number of arrival cells at β_0 is bounded as follows, $L \leq N + K$, where $N = \kappa(\beta_0)$ is the self-intersection degree of β_0 and K is the number of β_0 -remote BTS-instants performed by the walk. Then we can write that $K = K_1 + K_2 + K^{(3)} + K^{(4)}$, where

- K_1 is the number of BTS-instants performed by the μ_1 -edges or by the p -edges attached to them (with no respect to the orientation of p -edges);
- K_2 is the number of BTS-instants performed by the μ_2 -edges or by the p -edges attached to them;
- $K^{(3)}$ is the number of BTS-instants performed by the μ_m -edges, $m \geq 3$ or by the p -edges attached to them;
- $K^{(4)}$ is the number of BTS-instant performed by the ν -edges.

Next, we split $K_1 = K'_1 + K''_1$, where K'_1 is the number of BTS-instants performed by the double edges. We also consider $K_2 = K'_2 + K''_2$, where K'_2 is the number of the BTS-instants performed by the μ_2 -edges without p -edges attached to them.

It is clear that the number of simple BTS-instants K'_2 is bounded by the number of open simple self-intersections r , $K'_2 \leq r$. Also we can write that

$$K'_1 \leq K''_1 + K_2 + K^{(3)} + K^{(4)}. \quad (4.11)$$

We prove this inequality in subsection 6.4. Then we get the following lower bound $K''_1 + r + K''_2 + K^{(3)} + K^{(4)} \geq K/2$.

It is easy to see that the number of μ_1 -edges that have a multiple p -edge attached is not less than the number

$$R_1 = (\lfloor K''_1/(2k_0) \rfloor + 1) \cdot I_{\{K''_1 > 0\}},$$

where $I_{\{K_1'' > 0\}}$ is equal to 1 if $K_1'' > 0$ and to zero otherwise. The number of μ_2 -vertices such that the corresponding edges have p -edges attached is not less than $R_2 = (\lfloor K_2''/(4k_0) \rfloor + 1) I_{\{K_2'' > 0\}}$. Regarding the μ_m -vertices with p -edges attached, we see that their number is not less than $R_m^{(3)} = (\lfloor K_m^{(3)}/(2k_0 m) \rfloor + 1) I_{\{K_m^{(3)} > 0\}}$, $3 \leq m \leq k_0$ where $K_m^{(3)}$ is the number of BTS-instants performed by the edges of the corresponding degree of μ -self-intersection of by the p -edges attached.

We split the estimate of $Z_{2s}^{(3,1)}$ into the following four stages.

Part I. Let us start our estimates with the factor that correspond to the ν -edges and the BTS-instants at them. There are $K^{(4)}$ BTS-instants that cannot be the first arrival instants at these vertices. Then for the corresponding factor we can write the following estimate

$$F_{2s}^{(\bar{\nu})} \leq \sum_{r_{k_0+1} + \dots + r_s = K^{(4)}} \prod_{k=k_0+1}^s \binom{(k-1)\nu_k}{r_k} \frac{1}{\nu_k!} \left(n \frac{(2C_1 s U_n^2)^k}{v^{2k} \rho^k} \right)^{\nu_k}, \quad (4.12)$$

where the sum runs over $r_j \geq 0$. Denoting $|\bar{\nu}|_1 = \sum_{k \geq k_0+1} (k-1)\nu_k$ and taking into account inequality

$$\binom{(k-1)\nu_k}{r_k} \leq \frac{((k-1)\nu_k)^{r_k}}{r_k!},$$

we deduce from (4.12) that

$$\begin{aligned} \sum_{\bar{\nu}}^* F_{2s}^{(\bar{\nu})} &\leq \frac{1}{h^{K^{(4)}}} \sum_{\sigma_\nu \geq 0}^* \sum_{\bar{\nu}: |\bar{\nu}|_1 = \sigma_\nu} \frac{(h\sigma_\nu)^{K^{(4)}}}{K^{(4)}!} \cdot \frac{1}{(\log n)^{\sigma_\nu}} \\ &\quad \times \prod_{k=k_0+1}^s \left(n \frac{(2C_1 s U_n^2)^k}{v^{2k} \rho^k} (\log n)^{k-1} \right)^{\nu_k}, \end{aligned}$$

where the stars mean that the sum over $\bar{\nu}$ runs over such values of $\nu_k \geq 0$ that $\sum_{k \geq K_0+1} k\nu_k \geq K^{(4)}$. Let us also note that if $K^{(4)} = 0$, then the sum over σ_ν starts by 0, and if $K^{(4)} \geq 1$, then the sum starts by k_0 .

Taking into account that $U_n^2 = n^{2\delta}$ with the choice of (4.6), we see that

$$\sum_{\bar{\nu}}^* F_{2s}^{(\bar{\nu})} \leq \frac{1}{h^{K^{(4)}}} \sum_{\sigma_\nu \geq 0} \left(\frac{h}{\log n} \right)^{\sigma_\nu} \exp \left\{ \tilde{A}_n^{(k_0)} \right\},$$

where $\tilde{A}_n^{(k_0)} = n(\log n)^{k_0-1} (2C_1 s U_n^2)^{k_0} (v^2 \rho)^{-k_0}$ is such that $\tilde{A}_n^{(k_0)} = o(1)$ as $n \rightarrow \infty$ (cf. (4.7)). This gives the following bound

$$\sum_{\bar{\nu}}^* F_{2s}^{(\bar{\nu})} \leq \frac{1}{h^{K^{(4)}}} e^{\tilde{A}_n^{(k_0)}}. \quad (4.13)$$

Part II. Regarding the μ_m -vertices, we observe that their total number is not less than $R^{(3)} = (\lfloor K^{(3)}/(2k_0^2) \rfloor + 1) I_{\{K^{(3)} > 0\}}$. Then the corresponding factor is

estimated as follows,

$$F_{2s}^{\bar{\mu}^{(3)}} \leq \frac{1}{h^{R^{(3)}}} \sum_{\bar{\mu}^{(3)}}^* \frac{1}{\mu_3!} \left(\frac{6s^3 h}{n^2} \right)^{\mu_3} \prod_{m=4}^{k_0} \frac{1}{\mu_m!} \left(\frac{(2k_0)^m s^m h}{n^{m-1}} \right)^{\mu_m},$$

where the star means that the sum over $\bar{\mu}^{(3)} = (\mu_3, \dots, \mu_{k_0})$, $\mu_m \geq 0$ is such that $\sum_{m \geq 3} m \mu_m \geq R^{(3)}$. Then we get the following upper bound

$$\sum_{\bar{\mu}^{(3)}}^* F_{2s}^{\bar{\mu}^{(3)}} \leq \frac{1}{h^{R^{(3)}}} \exp\{6\chi h(1 + o(1))\}. \quad (4.14)$$

Part III. Regarding the set of $p_1, p_2, \dots, p_{2k_0-1}$ simple and multiple p -edges, we assume that $p_1'', p_2'', \dots, p_{2k_0-1}''$ of them are attributed to $\mu^{(3)}$ -edges and the remaining $p_1', p_2', \dots, p_{2k_0-1}'$ edges are attributed to μ_1 -edges and to μ_2 -edges.

Let r and R_2 vertices are chose among μ_2 vertices. The corresponding factor that contributes to the estimate of the contribution of such walks is bounded by

$$\frac{1}{h^r} \cdot \frac{\mu_2!}{(\mu_2 - r - R_2)! r! R_2!} \left(\frac{\tilde{s}^2}{2n} \right)^{\mu_2 - r - R_2} \left(\frac{6sh\theta^*}{n} \right)^r \left(\frac{s^2}{2n} \right)^{R_2}. \quad (4.15)$$

Denoting $p_j' = a_j + b_j$, we see that the number of possibilities to distribute a_1, \dots, a_{2k_0-1} edges of p_1' simple p -edges, p_2' double p -edges and p_{2k_0-1}' edges over μ_1 -edges is given by the expression

$$\frac{\mu_1!}{(\mu_1 - a_1 - \dots - a_{2k_0-1})! a_1! \dots a_{2k_0-1}!} \leq \prod_{l=1}^{2k_0-1} \frac{\mu_1^{a_l}}{a_l!} \leq \prod_{l=1}^{2k_0-1} \frac{s^{a_l}}{a_l!}.$$

Then the corresponding factor is bounded by

$$\prod_{l=1}^{2k_0-1} \frac{s^{a_l}}{a_l!} \left(\frac{d^l}{\rho^l} \tilde{V}_{2l+2} \right)^{a_l}. \quad (4.16)$$

Regarding the distribution of $b_1, b_2, \dots, b_{2k_0-1}$ simple and multiple p -edges such that $b_1 + \dots + b_{2k_0-1} = R_2$ over R_2 μ_2 -edges and taking into account expression (4.16), we can write the following expression that estimates the corresponding factor,

$$\begin{aligned} & \frac{R_2!}{h^{R_1+R_2}} \sum_{a_1+b_1=p_1'} \frac{1}{a_1!} \left(\frac{sd\tilde{V}_4}{\rho} \right)^{a_1} \cdot \frac{1}{b_1!} \left(\frac{hd\tilde{V}_4}{\rho} \right)^{b_1} \\ & \times \prod_{l=2}^{2k_0-1} \sum_{a_l+b_l=p_l'} \frac{1}{a_l!} \left(\frac{hsd^l}{\rho^l} \tilde{V}_{2l+1} \right)^{a_l} \frac{1}{b_l!} \left(\frac{hd^l}{\rho^l} \tilde{V}_{2l+2} \right)^{b_l}, \end{aligned} \quad (4.17)$$

where $a_2 + a_3 + \dots + a_{2k_0-1} = R_1$ and $h \geq 1$.

Combining (4.15) with (4.17), we get the bound for the factor that corresponds to μ_1 and μ_2 -edges with p -edges attached,

$$\begin{aligned} & \frac{1}{h^{r+R_1+R_2}} \cdot \frac{\mu_2!}{(\mu_2 - r - R_2)! r!} \cdot \left(\frac{\tilde{s}^2}{2n} \right)^{\mu_2 - r - R_2} \left(\frac{6sh\theta^*}{n} \right)^r \\ & \times \frac{1}{p_1'!} \left(\frac{sd\tilde{V}_4}{\rho} + \frac{hs^2d\tilde{V}_4}{2n\rho} \right)^{p_1'} \cdot \prod_{l=2}^{2k_0-1} \frac{1}{p_l'!} \left(\frac{hsd^l}{\rho^l} \tilde{V}_{2l+2} \left(1 + \frac{s}{n} \right) \right)^{p_l'}, \end{aligned} \quad (4.18)$$

where $\mu_2 \geq R_2$.

Part IV. Finally, it remains to take into account the distribution of $p_1'', \dots, p_{2k_0-1}''$ of p -edges over the $\mu^{(3)}$ -edges. This gives the factor

$$\prod_{l=1}^{2k_0-1} \frac{1}{p_l''!} \left(\frac{sd^l}{\rho^l} \tilde{V}_{2l} \right)^{p_l''}. \quad (4.19)$$

Now we are ready to complete the estimate of $Z_{2s}^{(3,1)}$. Remembering inequality (4.10), using the bounds (4.13), (4.14) and combining these with expressions (4.18) and (4.19), we get after summations over p_l' and p_l'' such that $p_l' + p_l'' = p_l$ the following upper bound

$$\begin{aligned} Z_{2s}^{(3,1)} & \leq n \hat{V}_2^s \sum_{\theta \in \Theta_{2s}} \sum_{\mu_2=0}^{C_0 s^2/n} \sum_{\bar{\mu}^{(3)}} \sum_{\bar{p}} \sum_{\bar{v}} \exp \left\{ -\frac{\tilde{s}^2}{2n} + \frac{s\sigma_\mu}{n} \right\} \\ & \times \sum_{d > n^\delta} \sum_{L=1}^s \sum_{N+K \geq L} \sum_{K_1''+r+K_2''+K^{(3)}+K^{(4)} \geq K/2} \frac{(d+L-1)!}{d! (L-1)!} e^{-\eta d} \cdot \Phi_N \\ & \times \sum_{\mu_2 \geq R_2} \sum_{r=0}^{\mu_2 - R_2} \frac{1}{h^{N+r+R_1+R_2+R^{(3)}+K^{(4)}}} \cdot \frac{\mu_2!}{(\mu_2 - r - R_2)! r!} \left(\frac{\tilde{s}^2}{2n} \right)^{\mu_2 - r - R_2} \\ & \times \left(\frac{6sh\theta^*}{n} \right)^r \exp \left\{ \frac{sd\tilde{V}_4}{\rho} + \frac{hs^2d\tilde{V}_4}{2n\rho} + \sum_{l=2}^{2k_0-1} \frac{sd^l \tilde{V}_{2l+2}}{\rho^l} (1+2h) \right\} \cdot e^{6\chi h + \tilde{A}_h^{(k_0)}}, \end{aligned} \quad (4.20)$$

where

$$\Phi_N = \begin{cases} h^N (2k_0)^N \frac{d^N \tilde{V}_{2N+2}}{\rho^N}, & \text{if } N \leq k_0; \\ h^N \frac{C_1^N s^N \tilde{V}_{2N+2}}{\tilde{V}_2^N \rho^N}, & \text{if } k_0 \geq k_0 + 1, \end{cases}$$

we used (4.18) under assumption that $h > 1$ and $s/n \leq 1$.

It is not hard to deduce from (4.20) that

$$\begin{aligned} Z_{2s}^{(3,1)} & \leq n \tau_s \hat{V}_2^s e^{(C_0+36h)\chi^3(1+o(1))} B_s(6\chi^{3/2}h) \\ & \times s^6 h \sum_{d > n^\epsilon} \sum_{L=1}^s \frac{(d+L-1)!}{d! (L-1)!} e^{-\eta d} \cdot h^{-L/(16k_0^2)}. \end{aligned} \quad (4.21)$$

Applying the Stirling's formula for the factorial terms, we see that

$$\frac{1}{\tilde{h}^L} \cdot \frac{(d+L-1)!}{d!(L-1)!} = \frac{e^x}{\tilde{h}^x} \sqrt{\frac{d+x}{2\pi dx}} \left(1 + \frac{d}{x}\right)^x (1 + o(1))$$

for large values of $L-1 = x$. Thus, the question is reduced to the analysis of the function

$$f(x) = \frac{1}{\tilde{h}^x} \left(1 + \frac{d}{x}\right)^x, \quad x \geq 1.$$

The function $g(x) = \ln f(x)$ has only one extremum point x_0 that verifies equation

$$\ln \left(1 + \frac{d}{x_0}\right) = \frac{d}{x_0 + d} + \ln \tilde{h}.$$

This equality implies that $x_0 \leq d(\tilde{h} - 1)^{-1}$ and therefore $f(x) \leq \exp\{d/\tilde{h}\}$ for all $x \in [1, +\infty)$. Taking $h = h^* = (2\eta)^{16k_0^2}$, we arrive at the final estimate

$$Z_{2s}^{(3,1)} \leq \frac{n}{4^s} \frac{(2s)!}{s!(s+1)!} \exp\{(C_0 + 36h^*)\chi^3(1 + o(1))\} B_s(6h^*\chi^{3/2}) s^8 e^{-\eta n^\epsilon/2}.$$

Then obviously, $Z_{2s_n}^{(3,1)} = o(1)$ as $n \rightarrow \infty$.

To estimate of $Z_{2s}^{(3,2)}$, one can repeat the arguments used to estimate $Z_{2s}^{(2,2)}$ and $Z_{2s}^{(3,1)}$. We omit the corresponding computations that show that $Z_{2s_n}^{(3,2)} = o(1)$ as $n \rightarrow \infty$.

Remembering inequality (4.4) for $Z_{2s}^{(1,1)}$ and combining it with the estimates of $Z_{2s}^{(1,2)}$, $Z_{2s}^{(2)}$, and $Z_{2s}^{(3)}$, it is easy to complete the proof of Theorem 3.1.

4.4 Proof of Theorem 2.1

Using the standard arguments of the probability theory, we can write that

$$\mathbf{P} \left\{ \hat{\lambda}_{\max}^{(n, \rho_n)} > 2v \left(1 + \frac{x}{n^{2/3}}\right) \right\} \leq \frac{\mathbf{E} \operatorname{Tr} (\hat{H}^{(n, \rho_n)})^{2s_n}}{(2v(1 + xn^{-2/3}))^{2s_n}}, \quad (4.22)$$

where $\hat{\lambda}_{\max}^{(n, \rho_n)}$ is the maximal in absolute value eigenvalue of $\hat{H}^{(n, \rho_n)}$. Regarding the results of previous subsections, we see that

$$\limsup_{n \rightarrow \infty} \mathbf{E} \operatorname{Tr} (\hat{H}^{(n, \rho_n)})^{2s_n} \leq \mathcal{L}(\chi) = \frac{1}{\sqrt{\pi\chi^3}} e^{(C_0+36)\chi^3} B(6\chi^{3/2}).$$

Then it follows from (4.22) that

$$\limsup_{n \rightarrow \infty} \mathbf{P} \left\{ \hat{\lambda}_{\max}^{(n)} > 2v \left(1 + \frac{x}{n^{2/3}}\right) \right\} \leq \inf_{\chi > 0} \mathcal{L}(\chi) e^{-x\chi} = \mathcal{G}(x). \quad (4.23)$$

Let us consider the subset $\Lambda_n = \cap_{1 \leq i < j \leq n} \{\omega : |a_{ij}| \leq U_n\}$. Denoting $\lambda_{\max}^{(n, \rho_n)} = \lambda_{\max}(H^{(n, \rho_n)})$, we can write that

$$\mathbf{P} \left\{ \lambda_{\max}^{(n, \rho_n)} > y \right\} = \mathbf{P} \left\{ \hat{\lambda}_{\max}^{(n, \rho_n)} > y \right\} + \mathbf{P} \left\{ (\lambda_{\max}^{(n, \rho_n)} > y) \cap \bar{\Lambda}_n \right\}. \quad (4.24)$$

Clearly,

$$\mathbf{P}\{\bar{\Lambda}_n\} \leq \sum_{1 \leq i < j \leq n} \mathbf{P}\{|a_{ij}| > U_n\} \leq n^2 \frac{\mathbf{E}|a_{ij}|^{12+2\phi}}{U_n^{12+2\phi}}. \quad (4.25)$$

Assuming that the moment $V_{2+2\phi}$ of a_{ij} exists, we see that the choice of δ in (4.6) such that $\delta > (6 + \phi)^{-1}$ is sufficient for the right-hand side of (4.25) to vanish in the limit $n \rightarrow \infty$. Relation $\varepsilon + \varepsilon_0 = 6\delta$ together with the condition $\varepsilon > \varepsilon_0$ implies that $2\varepsilon > 6/(6 + l_0)$. Therefore it is sufficient to take the value

$$\varepsilon_0 = \frac{3}{6 + \phi} \quad (4.26)$$

to have Theorem 3.1 true and to deduce from (4.22) and (4.24) the estimate

$$\limsup_{n \rightarrow \infty} \mathbf{P}\left\{\lambda_{\max}^{(n, \rho_n)} > 2v \left(1 + \frac{x}{n^{2/3}}\right)\right\} \leq \inf_{\chi > 0} \mathcal{L}(\chi) e^{-x\chi} = \mathcal{G}(x). \quad (4.27)$$

Elementary analysis show that $\mathcal{G}(x) \leq e^{-Cx^{3/2}}$. Theorem 2.1 is proved.

4.5 Proof of Theorem 2.2

Using the following representation of the moments of $\tilde{H}^{(n, \rho_n)}$,

$$\tilde{L}_{2s_n}^{(n, \rho_n)} = \mathbf{E} \operatorname{Tr} \left(\tilde{H}^{(n, \rho_n)} \right)^{2s_n} = \tilde{Z}_{2s_n}^{(1)} + \tilde{Z}_{2s_n}^{(2)} + \tilde{Z}_{2s_n}^{(3)}, \quad (4.28)$$

where $\tilde{Z}_{2s_n}^{(i)}$, $i = 1, 2, 3$ are determined in the same way as it is done for $L_{2s}^{(n, \rho_n)}$ at the beginning of this section, we see that the estimate of $\tilde{Z}_{2s_n}^{(1)}$ is the same as that of $Z_{2s_n}^{(1)}$ and (4.4) is true.

To get the estimate of $\tilde{Z}_{2s_n}^{(2)}$ we repeat the computations used to estimate $Z_{2s_n}^{(2)}$ with the only difference that (4.7) is replaced by expression

$$n \frac{(C_1 U^2 s)^k}{(v^2 \rho)^k} = \left(\frac{C_1 U^2}{v^2} \right)^k n^{1-2\varepsilon k_0/3-2\varepsilon(k-k_0)/3}, \quad k \geq k_0 + 1.$$

Then the choice of the technical value $k_0 = \lfloor \frac{3}{2\varepsilon} \rfloor + 1$ (cf. (4.6)) is sufficient to have $\tilde{Z}_{2s_n}^{(2)} = o(1)$ as $n \rightarrow \infty$. The same concerns the estimate of $\tilde{Z}_{2s_n}^{(3)}$.

These arguments show that

$$\limsup_{n \rightarrow \infty} \tilde{L}_{2s_n}^{(n, \rho_n)} \leq \frac{1}{\sqrt{\pi\chi^3}} B(6\chi^{3/2}) e^{(C_0+36)\chi^3} \quad (4.29)$$

for any positive $\varepsilon \in (0, 1/2]$. It is easy to see that (4.29) implies (2.5). Theorem 2.2 is proved.

5 Estimate from below

Let us consider random matrices $H^{(n, \rho)}$ (2.1) with random variables a_{ij} that have all moments finite. Then the following statement for the moments $L_n^{(n, \rho)}$ (2.6) is true.

Theorem 5.1. *Let $s_n = \chi n^{2/3}$ and $\rho_n = \sigma n^{2/3}$ with $\chi, \sigma > 0$. Then*

$$\liminf_{n \rightarrow \infty} L_{2s_n}^{(n, \rho_n)} \geq \frac{n}{4^s} \frac{(2s)!}{s!(s+1)!} \cdot \frac{16V_4 s_n}{\rho_n} = \frac{16V_4}{\sigma(\pi\chi)^{1/2}} (1 + o(1)), \quad (5.1)$$

where $V_4 = \mathbf{E}|a_{ij}|^4$ and $\mathbf{E}|a_{ij}|^2 = v^2 = 1/4$.

Proof.

Let us consider a Dyck path θ_{2s} and point out two marked instants of time t_1 and t_2 such that the corresponding vertices α and β of the Catalan tree $T(\theta_{2s})$ have the same parent. Let us denote the couple $\{\theta_{2s}, (t_1, t_2)\}$ by $\theta_{2s}^{(\alpha, \beta)}$. To show (5.1), we construct the set of all walks with μ_2 simple self-intersections $\mathcal{W}_{2s}^{(\mu_2)}$ that have a self-intersection (t_1, t_2) . It is sufficient to consider the walks that are of the tree-type structure, i.e. that have no BTS-instants.

It is clear that the weight of each walk $W_{2s} \in \mathcal{W}_{2s}^{(\mu_2)}$ is bounded from below by

$$\Pi(W_{2s}) \geq \frac{v^{2s-4} V_4}{n^{s-1} \rho}. \quad (5.2)$$

There are $s - \mu_2$ vertices in the graph $g(W)$ of such walk.

Assuming that μ_2 is such that $0 \leq \mu_2 \leq M = C_2 n^{1/3}$ with some constant C_2 , we conclude that the number of possibilities to construct μ_2 simple self-intersections on s vertices can be bounded from below as follows

$$\frac{1}{2^{\mu_2} \mu_2! (s - 2\mu_2)!} \geq \frac{1}{\mu_2!} \left(\frac{(s - 2M)^2}{2} \right)^{\mu_2}.$$

Then we can write the following inequality,

$$\begin{aligned} L_{2s}^{(n, \rho)} &\geq \sum_{\theta_{2s}^{(\alpha, \beta)}} \sum_{\mu_2=0}^M \sum_{I_{2s} \in \mathcal{C}(W_{2s})} \sum_{W_{2s} \in \mathcal{W}_{2s}^{(\mu_2)}} \Pi(W_{2s}) \\ &\geq \exp \left\{ -\frac{s^2}{2n} \right\} \sum_{\mu_2=0}^M \frac{1}{\mu_2!} \left(\frac{(s - M)^2}{2n} \right)^{\mu_2} \cdot \frac{nv^{2s-4} V_4}{\rho} \sum_{\theta_s^{(\alpha, \beta)}} 1, \end{aligned} \quad (5.3)$$

where we have used an elementary analog of the formula (4.2).

Clearly, the following sum vanishes for large n ,

$$\sum_{\mu_2 > M} \frac{1}{\mu_2!} \left(\frac{s^2}{2n} \right)^{\mu_2} \leq \frac{1}{\sqrt{2\pi C_2 n^{1/3}}} \sum_{\mu_2 \geq M} \left(\frac{es^2}{2C_2 n^{4/3}} \right)^{\mu_2} = o(1), \quad n \rightarrow \infty \quad (5.4)$$

and this is true for any positive C_2 .

It follows from Lemma 6.3 (see Section 6) that $\sum_{\theta(\alpha, \beta)} 1 = s\tau_s(1 + o(1))$ as $s \rightarrow \infty$. This result, combining with (5.3) and (5.4), gives, after elementary computations, the lower bound (5.1). Theorem 5.1 is proved.

6 Auxiliary statements

6.1 Proof of Lemma 3.1

Any Dyck path θ_{2s} generates an ordered sequence of s marked instants of time $1 = \xi_1 < \xi_2 < \dots < \xi_s \leq 2s - 1$ that we denote by $\Xi_s = (\xi_1, \dots, \xi_s)$. Given $\bar{\mu}, \bar{p}, \bar{\nu}$, we can consider a collection of blue, green and black boxes that are gathered into groups according to $\bar{\mu}, \bar{p}, \bar{\nu}$. We mean here that we have μ_m groups of m ordered blue boxes, p_l groups of l ordered green boxes and ν_k groups of k ordered black ones.

Let us consider the walks $W_{2s} \in \mathcal{C}_{d,k_0}^{(\theta)}(\bar{\mu}, \bar{p}, \bar{\nu})$ and see how many values from Ξ_s can be seen in the corresponding groups of boxes. Let us consider first the black ones. Any W_{2s} generates a partition of s elements of Ξ_s into $1 + \sum_{k \geq k_0+1} \nu_k$ subsets of k elements and the number of such partitions is bounded by

$$\frac{s!}{(s - \sum_{k \geq k_0+1} \nu_k)! ((k_0 + 1)!)^{\nu_{k_0+1}} \nu_{k_0+1}! \dots (s!)^{\nu_s} \nu_s!} \leq \prod_{k=k_0+1}^s \frac{1}{\nu_k!} \left(\frac{s^k}{k!} \right)^{\nu_k}. \quad (6.1)$$

For such a partition, let us denote by $\Xi_s^{(\zeta_\nu)} \subseteq \Xi_s$ the subset of ξ_j 's that fill ζ_ν black boxes, $\zeta_\nu = |\bar{\nu}|$. It is clear that given $\Xi_s^{(\zeta_\nu)}$, the blue boxes can get the values from the subset $\Xi_s \setminus \Xi_s^{(\zeta_\nu)}$ of cardinality $s - \zeta_\nu$.

Regarding $\zeta_\mu^{(3)} = \sum_{m=3}^{k_0} m \mu_m$ blue boxes and repeating the arguments used in (6.1), we get the following estimate for the number of choice of the subsets $\Xi_s^{(\zeta_\mu)}$,

$$\prod_{m=3}^{k_0} \frac{1}{\mu_m!} \left(\frac{(s - \zeta_\nu)^m}{m!} \right)^{\mu_m} \leq \prod_{m=3}^{k_0} \frac{1}{\mu_m!} \left(\frac{s^m}{m!} \right)^{\mu_m}. \quad (6.2)$$

Let us consider the values at the boxes of simple μ -self-intersections. For simplicity, let us assume that we have a family of walks \tilde{W}_{2s} with only one simple μ -self-intersection, $\mu_2 = 1$. For any set of integers x, y, z such that $x + y + z = \zeta_p$, the set \tilde{W}_{2s} is separated into disjoint subsets $\tilde{W}_{2s}(x, y, z)$, where $W_{2s} \in \tilde{W}_{2s}(x, y, z)$ is such that x green boxes get values ξ_j that are less than the value of the first arrival instant t_1 at the vertex β of simple self-intersection and y green boxes get the values greater than the first arrival instant at β but less than the second arrival instant t_2 . Then the number of possible values for the pair (c_1, c_2) that determines the simple self-intersection $(t_1, t_2) = (\xi_{c_1}, \xi_{c_2})$ is bounded by the following sum over c_1 and c_2

$$\sum_{1 \leq x < c_1 < c_1 + y < c_2 \leq s}^* 1 \leq \frac{(s - \zeta_p - \zeta_\nu - \zeta_\mu^{(3)})^2}{2} = \frac{\tilde{s}^2}{2}, \quad (6.3)$$

where the star means that $c_i \in \Xi_s \setminus (\Xi_s^{(\zeta_\nu)} \cup \Xi_s^{(\zeta_\mu)})$. Certainly, $\tilde{W}_{2s} = \sqcup_{x,y} \tilde{W}_{2s}(x, y, z)$ and this explains the presence of \tilde{s} in the first factor at the right-hand side of (3.5). If the self-intersection is the open one, then at the instant ξ_{c_2} there is not more than $2\theta(\xi_{c_2} - 1) \leq 2\theta^*$ open vertices to choose as the vertex of such self-intersection. Then we get not more than $2s\theta^*$ pairs (c_1, c_2) to fill the pair of blue boxes.

Regarding the general case of μ_2 simple self-intersections, we use the recurrence argument based on the estimate (6.3). Assuming that there are r open self-intersections among μ_2 simple ones, we obtain by the standard reasoning the first factor of the right-hand side of (3.11),

$$\frac{1}{(\mu_2 - r)! r!} (6s\theta^*)^r \left(\frac{\tilde{s}^2}{2}\right)^{\mu_2 - r}; \quad (6.4)$$

see [26] for more details.

It is clear that the walk W_{2s} is determined by its values at the marked and non-marked instants of time. The addition factor 3 in front of $s\theta^*$ arises due to the possibility to choose one of three possible vertices to perform a non-marked step after the walk reaches the vertex of the open self-intersection [23]. The total number of such choices at other vertices of self-intersections is bounded by $(2k_0)^{m\mu_m}$ for the μ -vertices and by $(2k)^{k\nu_k}$ for the ν -vertices [9]. One can think about a certain rule v that prescribes the choice of vertices where to go at the non-marked instants of time. So, the cardinality of the set $\Upsilon_{k_0}^{(r)}(\bar{\mu}, \bar{\nu})$ of such rules for the walks from the class $\mathcal{C}_{k_0}(\bar{\mu}, \bar{p}, \bar{\nu})$ with r open simple self-intersections is bounded as follows [9]

$$|\Upsilon_{k_0}^{(r)}(\bar{\mu}, \bar{\nu})| \leq 3^r \prod_{m=3}^{k_0} (2k_0)^{m\mu_m} \prod_{k=k_0+1}^s (2k)^{k\nu_k}. \quad (6.5)$$

Let us consider the number of possible values in the green boxes. In the next subsection we show that given the marked instants of time that correspond to μ - and ν -edges and having fixed a rule $v \in \Upsilon_{k_0}^{(r)}(\bar{\mu}, \bar{\nu})$, the exit clusters of vertices of the walk graphs are uniquely determined. Then regardless the orientation of p -edges, one can see not more than d^l values in the each of p_l ordered groups of green boxes. This gives the factor $d^{lp_l}/p_l!$.

The green boxes are distributed over $\zeta_\mu = \sum_{m=1}^{k_0} m\mu_m$ μ -edges. This can be made by

$$\frac{\zeta_\mu!}{(\zeta_\mu - \zeta_p)! p_1! p_2! \cdots p_{2k_0-1}!}$$

ways, that is bounded by $\prod_{l=1}^{2k_0-1} s^{p_l}/p_l!$. Using this expression together with estimates (6.1)-(6.5), we get the right-hand side of (3.11). Lemma 3.1 is proved.

6.2 Trees, forests and multiple edges

In this subsection we prove an estimate of the number of trees (forests) that improves the results of [8] and [9].

Lemma 6.2. *Consider the set $\mathcal{T}_s^{(q)}$ of trees constructed with the help of k edges on q roots. Then for all $q \geq 1$ and $s \geq 0$ the following bound is true,*

$$|\mathcal{T}_s^{(q)}| \leq q^2 (2/3)^q |\mathcal{T}_{s+q}|, \quad (6.6)$$

where $|\mathcal{T}_{s+q}| = \tau_{s+q}$ is the Catalan number.

Proof. Clearly,

$$|\mathcal{T}_s^{(q)}| = \sum_{u_1+u_2+\dots+u_q=s} \tau_{u_1} \cdots \tau_{u_{q-1}} \tau_{u_q}, \quad (6.7)$$

where the sum runs over integers $u_j \geq 0$. Using the fundamental recurrence relation

$$\tau_{s+1} = \sum_{j=0}^s \tau_j \tau_{s-j}, \quad s \geq 0, \quad \tau_0 = 1, \quad (6.8)$$

we can rewrite (6.7) in the following form,

$$\begin{aligned} |\mathcal{T}_s^{(q)}| &= \sum_{v=0}^s \sum_{u_1+u_2+\dots+u_{q-2}+v=s} \tau_{u_1} \cdots \tau_{u_{q-2}} \sum_{u_{q-1}+u_q=v} \tau_{u_{q-1}} \tau_{u_q} \\ &= \sum_{v=0}^s \sum_{u_1+u_2+\dots+u_{q-2}+v=s} \tau_{u_1} \cdots \tau_{u_{q-2}} t_{v+1} = |\mathcal{T}_{s+1}^{(q-1)}| - |\mathcal{T}_{s+1}^{(q-2)}|. \end{aligned}$$

Slightly changing variables, we get the following recurrence relation of the triangle form,

$$|\mathcal{T}_s^{(q-1)}| = |\mathcal{T}_{s-1}^{(q)}| + |\mathcal{T}_s^{(q-2)}|, \quad q \geq 3, \quad s \geq 1 \quad (6.9)$$

with obvious initial conditions $|\mathcal{T}_0^{(q)}| = 1$ for all $q \geq 1$, $|\mathcal{T}_s^{(1)}| = \tau_s$, and $|\mathcal{T}_s^{(2)}| = \tau_{s+1}$ for all $s \geq 0$.

Let us note that (6.9) can be also obtained with the help of the generating function $\mathcal{F}(\vartheta, \varsigma) = \sum_{q \geq 1} \sum_{s \geq 0} |\mathcal{T}_s^{(q)}| \cdot \vartheta^q \varsigma^s$ that is given by relation

$$\mathcal{F}(\vartheta, \varsigma) = \frac{\vartheta \varphi(\varsigma)}{1 - \vartheta \varphi(\varsigma)},$$

where $\varphi(\varsigma) = \sum_{s=0}^{\infty} \tau_s \varsigma^s$ is the generating function of the Catalan numbers. It follows from (6.8) such that $\varphi(\varsigma)$ verifies the following equation

$$\varphi(\varsigma) = 1 + \varsigma \varphi^2(\varsigma) \quad (6.10)$$

and therefore $\varphi(\varsigma) = (1 - \sqrt{1 - 4\varsigma})/2\varsigma$.

Using (6.9) and the explicit form of the Catalan numbers, it is easy to check by recurrence that (6.6) holds. To do this, it is sufficient to check first that (6.6) are true at the border lines $|\mathcal{T}_0^{(q)}|$, $|\mathcal{T}_s^{(1)}|$ and $|\mathcal{T}_s^{(2)}|$ and the to fill the lines $s + q = m$ from the upper part to the bottom part and pass from m to $m + 1$. Lemma 6.2 is proved.

Clearly, (6.6) implies that

$$|\mathcal{T}_k^{(q)}| \leq (3/4)^q \tau_{k+q}, \quad \text{for all } k \geq 0, \quad q \geq q_0, \quad (6.11)$$

where one can take $q_0 = 100$.

Let us consider a Catalan tree T_k and denote by $N^{(2)}(T_k)$ the number of possibilities to mark two edges of T_k that have the same parent vertex. Clearly, the sum $N_k^{(2)} = \sum_{T_k \in \mathcal{T}_k} N^{(2)}(T_k)$ represents the number of even closed walks of $2k$ steps whose graphs have exactly one double edge in the sense of definitions of Section 3 and that have no other self-intersections. For the reader's convenience, we present here the proof of the following statement.

Lemma 6.3 [9] *The number $N_s^{(2)}$ is bounded from below, $N_s^{(2)} \geq s\tau_s$; moreover, the following explicit expression,*

$$N_s^{(2)} = \tau_s \left(s - \frac{3s}{s+2} \right). \quad (6.12)$$

is true.

Proof. It is not hard to see that

$$N_s^{(2)} = \sum_{u+v_1+v_2+v_3=s-2} (2u+1) \tau_u \tau_{v_1} \tau_{v_2} \tau_{v_3}, \quad s \geq 3$$

and therefore the generating function $N^{(2)}(\varsigma) = \sum_{k \geq 0} N_k^{(2)} \varsigma^k$ with $N_0^{(2)} = N_1^{(2)} = 0$ is given by relation

$$N^{(2)}(\varsigma) = \varsigma^2 \varphi^3(\varsigma) (2\varsigma \varphi'(\varsigma) + \varphi(\varsigma)).$$

This equality implies equality

$$N^{(2)}(\varsigma) = \frac{1-3\varsigma}{\sqrt{1-4\varsigma}} + (2\varsigma-1)\varphi(\varsigma).$$

Then relation (6.10) implies (6.12).

More generally, the total number of possibilities to mark l edges that have the same parent vertex at the Catalan trees is determined as follows,

$$N_s^{(l)} = \sum_{u+v_1+\dots+v_{2l-1}=s-l} (2u+1) \tau_{v_1} \tau_{v_2} \dots \tau_{v_{2l-1}}$$

and therefore its generating function $N^{(l)}(\varsigma)$ is given by relation

$$N^{(l)}(\varsigma) = \varsigma^l \varphi^{2l-1}(\varsigma) (2/\sqrt{1-4\varsigma} - \varphi(\varsigma)).$$

Then it is not hard to show that

$$N_s^{(l)} \leq 2^l s \tau_s. \quad (6.13)$$

Finally, let us study the tree-type walks that have no self-intersections excepting those that produce double edges. Let us denote by $M_s^{(2)}$ the sum over even closed walks whose graphs have simple or double edges, where each simple edge produces a factor a and each double edge produces a factor b .

Lemma 6.4. *Generating function $\Psi(\varsigma) = \sum_{s \geq 0} M_s^{(2)} \varsigma^s$ is given by the following equation*

$$\Psi(\varsigma) = 1 + a\varsigma \Psi^2(\varsigma) + b\varsigma^2 \Psi^4(\varsigma). \quad (6.14)$$

Proof. One can use the standard recurrence procedure with respect to the first edge (ρ, α) of the walk; this recurrence is standard when regarding the Catalan trees or the tree like walks of more complicated structure. Then it is not hard to see that the following relation holds

$$M_s^{(2)} = a \sum_{v_1, v_2}^{s-1} M_{v_1}^{(2)} M_{v_2}^{(2)} + b \sum_{u_1, u_2, u_3, u_4}^{s-2} M_{u_1}^{(2)} M_{u_2}^{(2)} M_{u_3}^{(2)} M_{u_4}^{(2)}, \quad (6.15)$$

where the sum runs over non-negative values of v_i and u_j such that $v_1 + v_2 = s - 1$ and $u_1 + u_2 + u_3 + u_4 = s - 2$. It is easy to show that (6.15) implies (6.14). Lemma 6.4 is proved.

Regarding (6.14) in the limit of $b \rightarrow 0$, we see that $\Psi(\varsigma) = \varphi(\varsigma) + b\Phi(\varsigma) + o(b)$, we get equality

$$\Phi(\varsigma) = 2a\varsigma\varphi(\varsigma)\Phi(\varsigma) + \varsigma^2\varphi^4(\varsigma) \quad (6.16)$$

that can be also obtained with the help of recurrence relation for $N_s^{(2)}$ of the form of (6.15). Relation (6.15) leads to another one form of $N^{(2)}(\varsigma) = \Phi(\varsigma)|_{a=1}$ that is

$$N^{(2)}(\varsigma) = \frac{\varsigma^2\varphi^4(\varsigma)}{1 - \varphi^2(\varsigma)} = \sum_{m=0}^{\infty} \varsigma^2\varphi^{4+m}(\varsigma). \quad (6.17)$$

Elementary calculus shows that (6.12) verifies (6.17).

The last remark concerns the limiting expression for (6.14) when $b = \chi V_4 = 1$, $\chi \rightarrow \infty$ and $a = V_2 \rightarrow 0$. Then $\Psi(\varsigma) \rightarrow \psi(\varsigma)$, where

$$\psi(\varsigma) = 1 + \varsigma^2\psi^4(\varsigma) \quad (6.18)$$

is the generation function of the numbers of chronological runs over the Catalan trees $\hat{N}_s^{(2)}$, $s \geq 0$, where each edge is passed four times. The first six elements of this sequence are 1, 1, 4, 22, 116, 741. It would be interesting to get the explicit expression for this new sequence that can be regarded as a kind of Catalan-type numbers.

6.3 D-lemma

In this subsection we prove our main technical result that is much in the spirit of the conditional probability. It can be considered as an essential improvement of the statements used in [9, 21, 26]. Without this result it would be difficult to get the correct value of the critical exponent that we discuss in Section 7.

Lemma 6.5. *Consider a subset of walks $\mathcal{W}_{2s}^{(\tilde{t}_0; d)}$, $\delta \geq 2$ such that at the vertex $\beta_0 = W_{2s}(\xi_{\tilde{t}_0})$ there exists one cell having its proper exit cluster Δ with the number of edges $\|\Delta\| = d$. Then*

$$|\mathcal{W}_{2s}^{(\tilde{t}_0; d)}| \leq e^{-\eta d} |\mathcal{W}_{2s}|, \quad (6.19)$$

where $\eta = \log(4/3)$.

Proof. Given t' , let us consider a subset $\mathcal{W}_{2s}^{(\tilde{t}_0; d, t')} \subseteq \mathcal{W}_{2s}^{(\tilde{t}_0; d)}$ such that t' is the parent cell $c(t')$ of Δ at β_0 and take one element $W_{2s}^{(\tilde{t}_0; d, t')} \in \mathcal{W}_{2s}^{(\tilde{t}_0; d)}(t')$. We denote by $\check{\mathcal{W}}_{2s}^{(\tilde{t}_0; d, t')}$ the family of walks $\check{W}_{2s}^{(\tilde{t}_0; d, t')} \subset \mathcal{W}_{2s}^{(\tilde{t}_0; d, t')}$ that coincide with $W_{2s}^{(\tilde{t}_0; d, t')}$ during first t' steps, i.e. such that $\check{W}_{2s}^{(\tilde{t}_0; d, t')}(t) = W_{2s}^{(\tilde{t}_0; d, t')}(t)$ for all $t \in [0, t']$. We also introduce a family of walks $\check{W}_{2s}^{(t')} \in \mathcal{W}_{2s}$ that coincide with $W_{2s}^{(\tilde{t}_0; d, t')}$ during t' first steps. We are going to show that

$$|\check{\mathcal{W}}_{2s}^{(\tilde{t}_0; d, t')}| \leq e^{-\eta d} |\check{\mathcal{W}}_{2s}^{(t')}|. \quad (6.20)$$

Let us denote by $\theta^{(t')}$ the θ -structure of the sequence $\{W_{2s}^{(\tilde{t}_0; d, t')}(t), 0 \leq t \leq t'\}$ and consider the corresponding Catalan tree $T' = T^{(\theta^{(t')})}$. Let us denote by v' the vertex of T' that is determined by the instant t' during the chronological run over T' ;

$$v' = \mathcal{R}_{T'}(t')$$

and denote by $\mathcal{D}_{T'}(v')$ the set of vertices of the descending part of T' . On Figure 1, the Catalan structure $\theta^{(8)}$ of the walk W'_{16} is presented by the tree T' , with the descending part given by the vertices β_1, \dots, β_5 .

The remaining parts $\check{\theta}^{(d, t')}$ of θ -structures of the walks $W \in \check{\mathcal{W}}_{2s}^{(\tilde{t}_0; d, t')}$ are constructed as follows: the vertex v' gets the set of edges Δ and the remaining $s - \|T'\| - d$ edges are used to build all possible trees on the set of roots $\mathcal{D}_{T'}(v') \sqcup \mathcal{V}(\Delta)$. The remaining parts $\check{\theta}^{(t')}$ of θ -structures of the walks $W \in \check{\mathcal{W}}_{2s}^{(t')}$ are constructed as all possible trees of $S - \|T'\|$ edges on $\mathcal{D}_{T'}(v')$ roots.

We see that the difference between $\check{\theta}^{(d, t')}$ and $\theta^{(d, t')}$ is that in the first ensemble, d edges are removed from the build process and are attributed to the vertex v' to serve as the roots. Lemma 6.2 and its corollary (6.11) say that in this case,

$$|\mathcal{T}_{s - \|T'\| - d}^{(\|\mathcal{D}_{T'}(v') \sqcup \mathcal{V}(\Delta)\|)}| \leq e^{-\eta d} |\mathcal{T}_{s - \|T'\|}^{(\|\mathcal{D}_{T'}(v')\|)}|. \quad (6.21)$$

To complete the proof of (6.20), it remains to note that the choice of vertices for open self-intersections for the walks that have θ -structures $\check{\theta}^{(t')}$ is greater than that for the walks with θ -structures $\check{\theta}^{(d, t')}$ because the height of the corresponding Dyck paths is greater in the first case with respect to the second.

Clearly, relation (6.19) follows from (6.20). Lemma 6.5 is proved.

Remark. There are two possibilities concerning the cell $c(t')$:

- a) the instant t' is marked and then $c(t')$ is the primary cell at β_0 ;
- b) the instant t' is non-marked and $c(t')$ is the imported cell at β_0 .

The difference between these two cases with respect to the position of the vertex v' in T' is such that in the first case $\|T'\| = t'$ while in the second case $\|T'\| < t'$.

Corollary of Lemma 6.5. *Let $\mathcal{W}^{(\tilde{t}_0; L, \bar{d}_L)}$ be the set of walks such that at the vertex β_0 there are L cells parent for the exit clusters Δ_j , $1 \leq j \leq L$ with $\|\Delta_j\| = d_j \geq 2$ and $(d_1, d_2, \dots, d_L) = \bar{d}_L$. Then*

$$|\mathcal{W}_{2s}^{(\tilde{t}_0; L, \bar{d}_L)}| \leq e^{-\eta D} |\mathcal{W}_{2s}|, \quad (6.22)$$

where $D = \sum_{j=1}^L d_j$.

This statement is proved by the recurrence procedure.

6.4 Short BTS-instants in double edges

In this subsection we prove that the number of the short BTS-instants at the double edges (SBTS-instants) is bounded from above by the total number of all other BTS-instants. To do this, we show that each SBTS-instant is to be preceded by another one BTS-instant and that each SBTS-instant at the time t' can make possible for the walk to perform another one SBTS-instant at the time $t'' > t'$ and the only one provided the time interval $[t' + 1, t'' - 1]$ contains no other BTS-instants.

Let us first observe that the edge $e_3 = (\alpha, \beta)$ corresponding to the step $(t - 1, t)$ of the walk can represent a short BTS-instant at the double edge if there exists the p -edge $e_1 = (\alpha, \beta)$ closed by a non-marked edge e'_1 and that there exists an edge e_2 attached to β that is t -open. It is not hard to see that this is possible only in the case when $e'_1 = (\alpha, \beta)$ and $e_2 = (\beta, \gamma)$ with some $\gamma \neq \beta$ since the graph of the walk has no loops.

Clearly, to create the couple of edges (e_1, e'_1) , the walk has to perform a BTS-instant at the time interval $[t + 1, t' - 1]$. Thus, the SBTS-instant is impossible to be created without a BTS-instant performed before. By recurrence, we see that there has to be the starting point for the possible sequence of SBTS-instants, and this starting point cannot be the SBTS-instant by itself.

The SBTS-instant (e_3, e'_2) closes the edge e_2 and then the walk performs a number of steps. If there will be another one SBTS-instant, then the walk should prepare a new couple of edges $(\tilde{e}'_2, \tilde{e}_3)$ that are non-marked and marked non-closed ones, respectively. Therefore the time interval from e'_3 till \tilde{e}_2 should contain the non-marked edges only. In the opposite case when the walk will perform a marked step, it will meet another one non-closed marked edge and this will give the open self-intersection that we prohibit. Therefore, a SBTS-instant can create only one couple $(\tilde{e}'_2, \tilde{e}_3)$ because any other consequent couple will be created with the help of open self-intersection.

7 Discussion

In present paper we have studied the asymptotic properties of the probability distribution of the spectral norm of large dilute random matrices. We have shown that the probability distribution of the maximal eigenvalue of dilute Wigner random matrices $H^{(n, \rho_n)}$, when regarded at the scale $n^{-2/3}$, admits the universal upper bound in the limit of infinite n, ρ_n such that $n^{2/3} \ll \rho_n \leq n$. This result is a consequence of the existence of a universal upper bound of the moments $L_{2s_n}^{(n, \rho_n)}$, $s_n = \chi n^{2/3}$ of $H^{(n, \rho_n)}$ by itself or the moments of corresponding random matrices with truncated elements (see Theorem 3.1, Section 3).

According to the general scheme developed in papers [23, 26], this kind of the asymptotic behavior of the moments $L_{2s_n}^{(n, \rho_n)}$ can be regarded as the strong evidence of the universality of the probability distribution of one (or several) maximal eigenvalues of $H^{(n, \rho_n)}$ in the limit of infinite n and $\rho_n = n^{2/3(1+\varepsilon)}$, $\varepsilon > 0$. Therefore one can expect that the dilute random matrices in the limit of the weak dilution, i.e. for the values of ρ_n such that $n^{2/3} \ll \rho_n \leq n$, $n \rightarrow \infty$ belong to the class of universality

determined by the Gaussian Orthogonal Ensemble of random matrices (GOE) in the case of real symmetric $H^{(n,\rho_n)}$, or by GUE in the hermitian case [16].

From the other hand, the estimate from below given by Theorem 5.1 shows that in the asymptotic regime $n, \rho_n \rightarrow \infty$ such that $\rho_n = O(n^{2/3})$, the estimate from below of $L_{2s_n}^{(n,\rho_n)}$ involves the factor V_4 that therefore should be present in the corresponding estimate from above, if it exists. This means that in the asymptotic regimes of the moderate and strong dilution, when $\rho_n = \sigma^{2/3}$ or $\rho_n = o(n^{2/3})$, respectively, the standard universality conjecture fails in the sense that the limiting expressions depend on the moments higher than V_2 of the random variables a_{ij} . As a consequence, the left-hand sides of (2.3) and (2.4) should involve expressions that depend on V_4 .

Moreover, the results of Section 5 show that in the asymptotic regime $n, \rho_n \rightarrow \infty$ when $\rho_n = \sigma n^\epsilon$ with $0 < \epsilon < 2/3$, to get the finite upper bound for the moment of the order s_n , one should restrict the growth of s_n and consider the case when s_n is proportional to ρ_n but not to $n^{2/3}$ as before. Then one can conclude that the scale at the border of the limiting spectrum of $H^{(n,\rho_n)}$ should be also changed to be proportional to ρ_n .

Therefore we can put forward a conjecture that the rate $\rho_n = n^{2/3}$ represents the critical point where the eigenvalue distribution at the edge of the spectrum changes its properties, in particular the limiting scale. One of the possible explanation of the role of the exponent $2/3$ in these studies could be obtained from the point of view of random graphs. A number of properties of the Erdős-Rényi random graphs change when the rate of ρ_n is changing [2, 7]. Thus it would be interesting to develop further the connections between the properties of random graphs and the spectral theory of random matrices and to interpret our results from the point of view of random graphs.

It is worth noting that in the asymptotic regime of strong dilution when $\rho_n = n^\epsilon$ with $(0, 2/3)$, the leading contribution to the moment $L_{2s_n}^{(n,\rho_n)}$ come from the walks of the tree-type structure, i.e. without the BTS-instants. This implies that the difference between the real symmetric and hermitian cases disappears with respect to the local properties of eigenvalues at the spectral edge. Moreover, only the walks with simple self-intersections survives in this limit that makes the studies more easy than before.

The estimates (6.12) and (6.13) for the number of the tree-type walks with multiple edges show that the contributions of terms $Z_{2s}^{(l)}, l = 2, 3$ used in Section 4 are overestimated in this case. Instead, one could expect to get the estimates of the form

$$L_{2s_n}^{(n,\rho_n)} \leq \frac{n\tau_s}{4^s} \exp \left\{ \frac{a_1 s_n}{\rho_n} V_4 + \frac{a_2 s_n}{\rho_n^2} V_6 + \frac{a_3 s_n}{\rho_n^3} V_8 + \dots \right\}, \quad (7.1)$$

where a_l are some constants. This inequality is much in the spirit of the results of [8], where the tree-type walks has been studied in the limit $n, s_n \rightarrow \infty$ but in the different asymptotic regime $\rho_n = (\log n)^{1+\epsilon}$.

Relation (7.1) indicates that in the limit of strong dilution, there can exist an asymptotic regime such that $\rho_n \leq s_n \ll s_n^2$, when the leading term of $L_{2s_n}^{(n,\rho_n)}$ involves the fourth moment V_4 but is free from the higher moments V_6, V_8 , etcetera. In this case the leading contribution to $L_{2s_n}^{(n,\rho_n)}$ can be related with the numbers $M_s^{(2)}$

introduced and studied in Section 6, Lemma 6.4.

It would be interesting to check whether this new type of universality really occurs in dilute random matrices. The method proposed in [8] to study the high moment of strongly dilute matrices based on the properties of the tree-type trajectories with multiple edges could be helpful in these studies.

References

- [1] Z. D. Bai and Y. Q. Yin, Necessary and sufficient conditions for the almost sure convergence of the largest eigenvalue of Wigner matrices, *Ann. Probab.* **16** 1729-1741 (1988)
- [2] B. Bollobás, *Random graphs*, Academic Press, Inc. London, (1985) 447 pp.
- [3] O. N. Feldheim and S. Sodin, A universality result for the smallest eigenvalues of certain sample covariance matrices, *Geom. Funct. Anal.* **20** (2010) 88123
- [4] Z. Füredi and J. Komlós, The eigenvalues of random symmetric matrices, *Combinatorica*, **1** (1981) 233-241
- [5] S. Geman, A limit theorem for the norm of random matrices, *Ann. Probab.* **8** (1980) 252-261
- [6] V.L. Girko, *Spectral properties of random matrices*, Nauka, Moscow (1988) (in Russian)
- [7] S. Janson, T. Łuczak, A. Ruciński, *Random Graphs*, J. Wiley& Sons, New York (2000)
- [8] A. Khorunzhiy, Sparse random matrices: spectral edge and statistics of rooted trees, *Adv. Appl. Probab.* **33** (2001) 124-140
- [9] O. Khorunzhiy, High moments of large Wigner random matrices and asymptotic properties of the spectral norm, *Preprint* <http://arxiv.org/abs/0907.3743>, to appear in *Random Operators/Stochastic Eqs.*
- [10] O. Khorunzhiy, A class of even walks and divergence of high moments of large Wigner random matrices, *Preprint* <http://arxiv.org/abs/1005.3231>
- [11] A. Khorunzhiy, B. Khoruzhenko, L. Pastur, and M. Shcherbina, The large-n limit in statistical mechanics and the spectral theory of disordered systems, in: *Phase Transitions and Critical Phenomena* **Vol. 15**, pp. 74-239, Academic Press, London, 1992
- [12] O. Khorunzhiy, W. Kirsch, and P. Müller, Lifshitz tails for spectra of Erdős-Rényi random graphs, *Ann. Appl. Probab.* **16** (2006) 295309
- [13] O. Khorunzhiy and J.-F. Marckert, Uniform bounds for exponential moment of maximum of a Dyck path, *Electr. Commun. Probab.* **14** (2009) 327-333
- [14] O. Khorunzhiy and V. Vengerovsky, Even walks and estimates of high moments of large Wigner random matrices, *Preprint* <http://arxiv.org/abs/0806.0157>

- [15] V.A. Marchenko and L.A. Pastur, Distribution of eigenvalues for some sets of random matrices, *Mathematics of the USSR-Sbornik* **1** (1967) (4): 457-483
- [16] M.L. Mehta, *Random Matrices* (2004) Amsterdam: Elsevier/Academic Press
- [17] A.D. Mirlin and Ya.V. Fyodorov, Universality of level correlation function of sparse random matrices, *J. Phys. A* **24** (1991) 2273-2286
- [18] L. Pastur, On the spectrum of random matrices, *Theor. Mathem. Physics* **10** (1972) 102-111
- [19] C. Porter (ed.), *Statistical Theories of Spectra: Fluctuations* (1965) Av-cad. Press, New-York
- [20] G.J. Rodgers and A.J. Bray, Density of states of a sparse random matrix, *Phys. Rev. B* **37** (1988) 3557-3562
- [21] A. Ruzmaikina, Universality of the edge distribution of the eigenvalues of Wigner random matrices with polynomially decaying distributions of entries, *Commun. Math. Phys.* **261** (2006) 277-296
- [22] Ya. Sinai and A. Soshnikov, Central limit theorem for traces of large symmetric matrices with independent matrix elements, *Bol. Soc. Brazil. Mat.* **29** (1998) 1-24
- [23] Ya. Sinai and A. Soshnikov, A refinement of Wigner's semicircle law in a neighborhood of the spectrum edge for random symmetric matrices, *Func. Anal. Appl.* **32** (1998) 114-131
- [24] S. Sodin, Random matrices, non-backtracking walks, and the orthogonal polynomials, *J. Math. Phys.* **48** (2007) 123503
- [25] S. Sodin, The Tracy-Widom law for some sparse random matrices, *Journal of Statistical Physics* **136** (2009) 834-841
- [26] A. Soshnikov, Universality at the edge of the spectrum in Wigner random matrices, *Commun. Math. Phys.* **207** (1999) 697-733
- [27] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, *Ann. Math.* **62** (1955) 548-564